Continuous Compounding and Annualization

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Contents

1 Introduction

In our discussion of highway maintenance we will discuss *optimal* maintenance policies, in which we choose the best interval to resurface the roads. This involves maintenance costs spread out over time and we know, in principle, how to handle them: we convert everything to present value. However, if we want to attack the optimality problem using calculus, then the discrete-time methods developed in (eg) CRP 763 won't quite do. The reason is that if we consider, say, a 30-year policy and then think of extending it one year, we get a jump discontinuity in the result, and this makes calculus effectively unusable.

The solution is to consider time as varying continuously, and this note discusses the analytics of continuous-time discounting. A related problem is that we want to consider a single representative year of the maintenance policy, and this requires that we "annualize" the relevant present value — that is, convert it to an equivalent once-a-year quantity. A final problem is that the particular sequence of maintenance costs we shall be concerned with is special: it consists of costs which are incurred periodically, once every so-many years (that is, the yerars in which we actually perform the maintenance). The final section considers this special problem.

2 Continuous Compounding

Suppose you deposit $P(0)$ today (at time 0) in a bank. At the end of 1 year the bank pays you interest, calculating it at an annual rate of r. Then at the end of one year, your $P(0)$ has become

$$P(0)(1+r)$$

Now suppose that the bank pays you interest twice a year, a six-month intervals. If r is the annual rate, then it will compute each payment based on a rate of $r/2$, and there will be two payments. So at the end of year 1 $P(0)$ will become

$$\begin{aligned} S(1,2) &= \left[P(0)\left(1+\frac{r}{2}\right)\right]\left(1+\frac{r}{2}\right) \\ &= P(0)\left(1+\frac{r}{2}\right)^2 \end{aligned}$$

where in $S(1,2)$ the first argument (1) indicates that this is at the end of 1 year, and the second (2) indicates that compounding is done twice a year.

What about three times a year? The effective interest rate is $r/3$, and there are three compounding periods, so at the end of the year $P(0)$ becomes

$$\begin{aligned} S(1,3) &= P(0)\left(1+\frac{r}{3}\right)\left(1+\frac{r}{3}\right)\left(1+\frac{r}{3}\right) \\ &= P(0)\left(1+\frac{r}{3}\right)^3 . \end{aligned}$$

We can see where this is going. If payments are compounded k times a year, then at the end of 1 year $P(0)$ becomes

$$S(1,k) = P(0)\left(1+\frac{r}{k}\right)^k .$$

If you leave your money on deposit for t years with compounding k times a year there will be kt separate compoundings, so at the end of t years you will have

$$S(t, k) = P(0)\left(1 + \frac{r}{k}\right)^{kt}$$

Note that $S(t, k)$ is a period-t quantity.

Now suppose that we allow the number of annual compoundings k to become larger and larger. In the limit we arrive at "continuous compounding" — in effect, the bank is making your interest payments "all the time". Of course, the "effective rate" per payment — the analog of r/k — gets smaller and smaller; but the number of payments gets larger and larger. What will $P(0)$ become the end of t years? To study this we need to look at what happens as the number of annual compoundings k tends to infinity, that is:

$$\begin{aligned} S(t, \infty) &= \lim_{k\to\infty} P(0)\left(1 + \frac{r}{k}\right)^{kt} \\ &= P(0) \lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^{kt} \end{aligned}$$

If we write $x = r/k$ then we can write the limit expression as

$$\lim_{k\to\infty}\left((1 + x)^{1/x}\right)^{rt}$$

(since $\left((1 + x)^{1/x}\right)^{rt} = (1 + x)^{rt/x} = (1 + x)^{rt/(r/k)} = (1 + x)^{kt}$). Since $x = r/k$ then $k \to \infty$ means that $x \to 0$. Note that the outer exponent doesn't depend on k, so we can write this as

$$\left(\lim_{x\to 0}(1 + x)^{1/x}\right)^{rt}$$

From the Binomial Theorem the quantity $(1 + x)^{1/x}$ tends to e $(= 2.7183\ldots)$ as x tends to zero, and we have

$$\lim_{x\to 0}\left((1 + x)^{1/x}\right)^{rt} = e^{rt}.$$

In other words, with continuous compounding, $P(0)$ will grow in t years to

$$S(t, \infty) = P(0)e^{rt}$$

3 Present Value with Continuous Compounding

Suppose that someone offers you $S(t)$ to be received in year t. To find the present value of this we need to find a quantity $P(0)$ such that you will be indifferent between receiving $P(0)$ *now* and $S(t)$ in year t. If your market opportunities are given by a banking system which compounds continuously at annual rate r, then at the end of t years your $P(0)$ will compound to

$$P(0)e^{rt}$$

In order for you to be indifferent between $S(t)$ in period t — note that since we are now assuming continuous compounding, this is the same as what we wrote as $S(t, \infty)$ in the last section — and receiving $P(0)$ *now* and leaving it on deposit until period t we must have

$$S(t) = P(0)e^{rt}$$

or, solving for the year-0 quantity, the present value:

$$P(0) = S(t)e^{-rt}.$$

In other words, with continuous compounding, *the present value of* $S(t)$ *received in year* t *is*

$$S(t)e^{-rt}$$

For reference, here is a table comparing the present value of $1 received at various times t and at various interest (discount) rates r, using continuous and discrete time compounding;

r	t	e^{-rt}	$\frac{1}{(1+r)^t}$
.03	10	.7408	.7441
	20	.5488	.5537
	30	.4066	.4120
.05	10	.6065	.6139
	20	.3679	.3769
	30	.2231	.2314
.08	10	.4492	.4632
	20	.2019	.2145
	30	.0967	.0994

As we can see, the results are generally quite close.

4 Annualization

Consider a project which has a single up-front (time-0) cost and a benefits extending through time. One way to evaluate the project is to convert the stream of benefits to its present value; then we can directly compare the two. But we sometimes want to think about what happens at each year of the project, and in this case we need to compare the annual benefit to some portion of the cost. To handle this, it is logical to think about reversing the idea of present value: that is, to construct a stream of costs which is equivalent to the original (time-0) cost. Since there are an infinite number of ways to do this, we shall also require that each of the costs in the constructed stream be the same: in other words we construct an annuity which is equivalent to the original time-0 cost. This process is known as annualization: it converts a single quantity to an equivalent annuity.

Suppose you invest $P(O)$ now in some (public) project that is projected to last T years. We seek an (annual) annuity amount A that is equivalent in present value to $P(0)$ now. With continuous compounding, we will need to add up the present value at each possible time between 0 and T. With continuous time, this means that the present value of our T-period annuity is an integral:

$$\begin{aligned} & A\int_{t=0}^{t=T} e^{-rt}\,dt \\ = & -A\frac{e^{-rT}-1}{r} \\ = & A\frac{1-e^{-rT}}{r} \end{aligned}$$

To find the annuity amount A which is equivalent to a project expenditure of $P(0)$ now, we must solve

$$P(0) = A\frac{1-e^{-rT}}{r}$$

for A, and the solution is

$$\begin{aligned} A &= \frac{rP(0)}{1-e^{-rT}} \\ &= P(0)\frac{r}{1-e^{-rT}} \end{aligned}$$

In other words, incurring a project cost of $P(0)$ now is equivalent to incurring a cost of

$$P(0)\frac{r}{1-e^{-rT}}$$

in each of years 0 to T. This is the annualized cost over a T year project life. If the project lasts forever, we need to see what happens as $T \to \infty$. By inspection, the term e^{-rT} tends to zero and the annualized cost over an infinite project lifetime is therefore:

$$P(0)r.$$

5 A Special Problem

In our discussion of road maintenance we will consider a special situation: a maintenance policy incurs a cost of C every T years starting in year T, and this pattern continues indefinitely. We want to find the annualized cost of the policy. We do this in two steps: first, we find the present value of the stream, and then we annualize that present value.

First, what is the present value of the stream? The first cost comes in year T; its present value is Ce^{-rT}. The second comes in year $2T$; its present value is Ce^{-2rT}. So we're looking at a series of present value terms like

$$\begin{aligned} & Ce^{-rT} + Ce^{-2rT} + Ce^{-3rT} + \dots \\ = \quad & C(e^{-rT} + e^{-2rT} + e^{-3rT} + \dots) \end{aligned}$$

where the series extends forever. We handle this as follows: we first compute the value of the series when it extends out for J terms, and then we take the limit as J tends to infinity.

Ignoring the constant C for the moment, our J-term series is

$$U = e^{-rT} + e^{-2rT} + e^{-3rT} + \dots + e^{-JrT}$$

We now use a trick very much like the one we use when summing a geometric series, except that this time we multiply each term by e^{-rT}. The result is

$$Ue^{-rT} = e^{-2rT} + e^{-3rT} + \dots + e^{-JrT} + e^{-(J+1)rT}.$$

Subtracting, we obtain

$$U - Ue^{-rT} = U(1 - e^{-rT}) = e^{-rT} - e^{-(J+1)rT}$$

(since everything in between cancels out), or

$$U = \frac{e^{-rT} - e^{-(J+1)rT}}{1 - e^{-rT}}.$$

Now, what happens as J tends to infinity? The second term in the numerator vanishes (tends to zero), and we have

$$U = \frac{e^{-rT}}{1 - e^{-rT}}.$$

We can make this look a bit neater by multiplying top and bottom by e^{rT} : the result is

$$U = \frac{1}{e^{rT} - 1}$$

and our conclusion is that the present value of in infinite stream of costs C incurred every T years is:

$$\frac{C}{e^{rT} - 1}$$

Finally we annualize this present value. Since the stream of costs continues indefinitely, the annualization is the one shown at the end of the last section, and we see that the annualized present value is

$$\frac{rC}{e^{rT} - 1}.$$