Understanding Horizontal and Vertical Addition

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This note walks you through some calculations involving both vertical and horizontal addition. It is basically a simple numerical example of the steers = beef+hides example we discussed in class.

- Setting: 2 individuals, 2 goods.
- Linear demands.
As you go through the analysis, you should ask yourself at each stage: what is the data, and what do I want to know from it?

- For horizontal addition, the data is a price, and what you want to know is the quantity demanded at that price.
- For vertical addition it’s the reverse: you’re given a quantity and you want to find the corresponding price (i.e., the average revenue) at that quantity.

The next slide illustrates the directional concepts inherent in horizontal and vertical addition.
\[ x = f(p) \]
\[ p = f^{-1}(x) \]
The First Good

- We denote the first good by $x_{11}$. The reason for the two subscripts is that we will be considering another good $x_{12}$ which will then be a part (joint product) of a composite good which we’ll call $x_1$. That’s why we have an additional level of subscripts.

- Individual 1’s demand for good 1: $x_{111} = 7 - 0.8p_1$.

- Individual 2’s demand for good 1: $x_{112} = 3 - 0.2p_1$. 
In this case we want to know the total demand for good 1 by the two individuals at a given price.

This is just what the demand functions tell us: we provide a price, and we can calculate the quantity demanded.

This is *horizontal addition* of the individual demand functions.

In this case, then:

\[ x_{11} = x_{111} + x_{112} 
= (7 - 0.8p_1) + (3 - 0.2p_1) 
= 7 - 0.8p_1 + 3 - 0.2p_1 
= 10 - 1p_1 \]

Check: at \( p_1 = 2 \), \( x_{111} = 7 - 1.6 = 5.4 \); \( x_{112} = 3 - 0.4 = 2.6 \), so \( x_{111} + x_{112} = 8 \); and \( x_{11} = 10 - 2 = 8 \).
The Second Good — Aggregate Demand

- We denote the second good by $x_{12}$.
  (Remember that this will be the second part of the composite good to be called $x_1$).
- Individual 1’s demand: $x_{121} = 6 - 1.5p_2$
- Individual 2’s demand: $x_{122} = 4 - 0.5p_2$
- Aggregate demand, by horizontal addition:

$$x_{12} = x_{121} + x_{122} = (6 - 1.5p_2) + (4 - 0.5p_2) = 10 - 2p_2$$
We have found, using horizontal addition, that the aggregate demands for the two goods are:

\[
\begin{align*}
\text{Good } x_{11} & : \quad 10 - p_1 \\
\text{Good } x_{12} & : \quad 10 - 2p_2
\end{align*}
\]
Joint Products

- Suppose now that $x_1$ and $x_2$ are joint products (effects) of a single good (activity) $x_1$; thus $x_1 = x_{11} + x_{12}$.
- We want to find the demand (average revenue) function for $x_1$.
- We are asking: given a quantity of $x_1$ (the entire, composite, product) what is the average revenue from that quantity?
- The technique needed to answer this is vertical addition.
- This means we want to add demands in the form: price = $f(\text{quantity})$.
- But our aggregate demands are not in that form: they tell us: quantity = $g(\text{price})$.
- So we have to invert the aggregate demands to get them into the appropriate forms for vertical addition. These are called the inverse demands.
Inverse Demands

To invert a demand function we solve for an independent variable as a function of the old dependent variable. In this case we solve for the own-price as a function of quantity.

For good $x_{11}$:

\[ x_{11} = 10 - p_1 \]
\[ p_1 = 10 - x_{11} \]

For good $x_{12}$:

\[ x_{12} = 10 - 2p_2 \]
\[ 2p_2 = 10 - x_{12} \]
\[ p_2 = 5 - 0.5x_{12} \]
We want to add these two demand vertically, i.e., at the same quantity of both goods.

So we re-write the inverse demands as functions of a common quantity $x_1$ corresponding to the “entire” (joint) effect:

\[ p_1 = 10 - x_1 \]
\[ p_2 = 5 - 0.5x_1 \]

So

\[ p = p_1 + p_2 \]
\[ = (10 - x_1) + (5 - 0.5x_1) \]
\[ = 15 - 1.5x_1 \]

gives the vertical sum of the two demand functions.
Suppose that the marginal cost of producing $x_1$ is also linear:

$$MC(x_1) = 2 + 1.8x_1$$

Then the equilibrium quantity equates demand and supply ($= MC$):

$$15 - 1.5x_1 = 2 + 1.8x_1$$
$$13 = 3.3x_1$$
$$x_1 = 13/3.3 = 3.9394$$

At this quantity, average revenue for $x_1$ is

$$15 - 1.5x_1 = 15 - (1.5 \times 3.9394) = 9.0909$$

Average revenue for $x_{11}$ is $10 - x_1 = 10 - 3.9394 = 6.0606$

Average revenue for $x_{12}$ is $5 - 0.5x_1 = 5 - (0.5 \times 3.9394) = 3.0303$

Note that the last two average revenues sum to $6.0606 + 3.0303 = 9.0909$
Market Failure

- Suppose the market for $x_{12}$ did not exist. Then equilibrium would be based on the (entire) marginal cost, but on the demand for the first “part” (product) $x_{11}$ only.
- We would have

$$10 - x_1 = 2 + 1.8x_1$$
$$8 = 2.8x_1$$
$$x_1 = \frac{8}{2.8} = 2.8571$$

- Note that the equilibrium output has fallen from 3.9394 to 2.8571 as our diagrams would indicate.
At this point we are in the familiar setting shown in the Figure.

The Pareto-Optimal demand (based on the vertically added demand function $D_1$) is $x_1^*$. 

When the market for $x_{12}$ does not exist, the observed output (based on $D_{11}$ alone) will be $x_1$. 

\[ S = MC \]
How can we bring about the correct level of output, in the presence of the market failure?

One way is present producers with a revised cost schedule, in order to induce them to undertake the Pareto-Optimal level of the activity.

In the figure, if the producers faced a (lower) marginal cost schedule $MC'$, they would produce the correct amount of output (based on $MC'$ and $D_{11}$).

So one solution is to set up a cost-side subsidy to mitigate the effects of the market failure.
How much subsidy? Since the correct output is where the vertically added demand intersects $MC$, we have (where $+$ means vertical addition):

$$D_{11} + D_{12} = MC$$

But in this case only $D_{11}$ is observed. So we re-arrange this to:

$$D_{11} = MC - D_{12} = MC'$$

This says that if producers faced a marginal cost schedule equal to $MC - D_{12}$ then they would choose the correct (socially desirable) level of output.
To find the subsidized marginal cost schedule we want to subtract $x_{12}$ (the average revenue for the unpriced output) from $MC$. Again, this involves vertical addition/subtraction.

So:

$$MC'(x_1) = MC(x_1) - x_{12}$$
$$= 2 + 1.8x_1 - (5 - 0.5x_1)$$
$$= -3 + 2.3x_1$$

This is the marginal cost schedule that we want the producers to use when making their decisions.
Will it work, in our numerical example?

- The individual now bases her actions on the average revenue for $x_{11}$ (the part for which market failure did not occur) and $MC'$.  
- Thus:

\[
\begin{align*}
10 - x_1 &= -3 + 2.3x_1 \\
13 &= 3.3x_1 \\
x_1 &= 13/3.3 = 3.9394
\end{align*}
\]

which is the Pareto-Optimal solution we found earlier.

- So the subsidy does indeed have the desired effect.