1. Consider the linear production function

\[ x = 2z_1 + 3z_2 + 6 \]

where \( x \) is the quantity of output producible from the input bundle \((z_1, z_2)\).

(a) Show that the marginal product of \( z_2 \) is 3. (Hint: suppose \( z_2 \) increases by \( \Delta \). Then \ldots)

(b) What would it mean for a marginal product to be negative? Can you think of a real-world example?

**Answer Sketch:** For the linear production function:

(a) The marginal product of a factor measures the increase in output brought about by an increase in one input, per unit input increase. So it is \([\text{change in output}] / [\text{change in an input}]\). Let the original output be \( x \). Now suppose utilization of \( z_2 \) increases by \( \Delta \). The new output is

\[ x' = 2z_1 + 3(z_2 + \Delta) + 6 \]

and the change in output is

\[ x' - x = (2z_1 + 3z_2 + 6) - (2z_1 + 3(z_2 + \Delta) + 6) = 3\Delta \]

The marginal product is the change in output per unit change in the input, so

\[ MP_{z_2} = \frac{x' - x}{\Delta} = \frac{3\Delta}{\Delta} = 3 \]

**Calculus remark:** for a general production function \( x = f(z_1, z_2, \ldots) \) the marginal product of input \( i \) is defined to be \( MP_i = \partial f / \partial z_i \). For our linear production function, the derivative is obviously 3.

(b) If the marginal product of a factor is negative, that would mean that increasing utilization of that factor (holding everything else constant) caused output to fall. It’s not exactly easy to think of examples, but here are some possibilities. First, agriculture, where the input in question is water. While usually watering your crops is a good thing, too much of it can drown them, which is a negative marginal product of water. If the inputs are labor, machines, and factory-floor space, you
can think of the labor force eventually getting so large relative to the machines and available space (that are both held constant) that people get in each others’ way, reducing productivity. A lot of people think that the administrative input to university production has long since reached the point of negative marginal products. (You hire more administrators and output — however measured — goes down).

2. (continuation). For the same linear production function,

(a) Suppose you are now using \((z_1, z_2) = (5, 4)\). Suppose you want to reduce your utilization of \(z_2\) by 1.5. What change must you make in \(z_1\) (assuming that you want output to be unchanged)?

(b) What can you say about returns to scale at the input bundle \((5, 4)\)?

**Answer Sketch:** Continuing our work with the linear production function,

(a) We shall find the quantity \(z_1\) that keeps us on the \(x\)-isoquant for any given amount of \(z_2\). We can do this by treating \(z_2\) and \(x\) as fixed parameters and solving for \(z_1\) and the result will of course depend on which levels of \(x\) and \(z_1\) we’re interested in. This is simple:

\[
\begin{align*}
x & = 2z_1 + 3z_2 + 6 \\
x - 3z_2 - 6 & = 2z_1 \\
z_1 & = \frac{x}{2} - 1.5z_2 - 3
\end{align*}
\]

In our case \(x = 28\) and \(z_2 = 2.5\), so

\[
\begin{align*}
z_1 & = 14 - (1.5 \times 2.5) - 3 \\
& = 7.25
\end{align*}
\]

So we will need to add \(7.25 - 5 = 2.5\) more units of \(z_1\) to the input bundle. You can quickly verify that at the bundle \((7.25, 2.5)\) we still produce \(x = 28\) units: just compute \((2\times 7.25)+(3\times 2.5)+6 = 28.0\).

**Calculus remark:** for the 2-input production function \(x = f(z_1, z_2)\), the slope is \(dz_2/dz_1 = -(\partial f/\partial z_1)/(\partial f/\partial z_2)\), which here is \(dz_2/dz_1 = -2/3.\) This tells us the rate at which \(z_2\) and \(z_1\) can substitute for one another, along an isquant. For the linear case this is exactly \(\Delta z_2/\Delta z_1 = -2/3.\) Then \(\Delta z_1 = (-3/2) \Delta z_2\) with \(\Delta z_2 = -1.5\) so \(\Delta z_2 = (-3/2) \times (-3/2) = 9/4 = 2.5\), as before.

(b) Returns to scale examines what happens to output following a proportionate change in _all_ inputs. So we just scale up all inputs by some \(\lambda > 1\): let’s take \(\lambda = 3\). Then we are looking at the input bundle \((15, 12)\), and output is

\[
x' = (2 \times 15) + (3 \times 12) + 6
\]

\[= 72\]

So scaling up all inputs by the factor \(\lambda = 3\) results in output increasing by the factor \(72/28. = 2.5714\) which is less than 3, so we have DRTS. It’s worth noting that different \(\lambda\)’s will in this case result in different _degrees_ of scale economies (ie the ratio \((x'/x)/\lambda\)), as you can verify, eg for \(\lambda = 2\).

3. Consider the linear demand function

\[
x = 16 - 2.1p_x + p_y + 0.001M
\]

where \(p_x\) is the own-price of \(x\), \(p_y\) is the price of some other good, and \(M\) is income. Suppose we currently observe \(p_x = 4.80\), \(p_y = 6.00\) and \(M = 40,000\).
(a) What is the own-price elasticity of demand? (Advice: be sure to count the zeros in the income coefficient).

(b) We know that a straight line has the same slope everywhere. Does this imply that the own-price elasticity would be unchanged if we started with (say) \( p_x = 3.00 \) (and \( p_y \) and \( M \) were as before)? Why or why not?

**Answer Sketch:** For this demand function:

(a) We start by computing demand at the given data. We have

\[
x = 16 - (2.1 \times 4.80) + 6 + (0.001 \times 40000) \\
= 51.92.
\]

At this point you can approach the elasticity question in two ways. First, you can recall that elasticity (call it \( \eta \)) is

\[
\eta = \frac{\Delta x \cdot p_x^0}{\Delta p_x \cdot x^0} \\
= \text{[slope]} \times \frac{p_x^0}{x^0}
\]

and that given linearity, the slope is just the coefficient of own price, namely \(-2.1\). So

\[
\eta = \frac{-2.1 \times 4.80}{51.92} \\
= -0.19414
\]

Second, you can do it by finite changes: generate some data at a nearby point and see what changes. Let’s look at \( p_x = 4.90 \). Then

\[
x' = 16 - (2.1 \times 4.90) + 6 + (0.001 \times 40000) \\
= 51.71
\]

Then, plugging in, we have

\[
\eta = \frac{51.71 - 51.92}{4.90 - 4.80} \cdot \frac{4.80}{51.92} \\
= \frac{-0.21 \times 4.80}{51.92} \\
= -0.19414
\]

as before.

(b) It is important to realize that a linear demand function has elasticities that differ at every point on the function. The reason goes back to the definition

\[
\eta = \frac{\Delta x \cdot p_x^0}{\Delta p_x \cdot x^0} \\
= \text{[slope]} \times \frac{p_x^0}{x^0}
\]

In this, the first term is certainly constant, but the second one isn’t. In fact, as we get closer and closer to the point where \( x = 0 \) (where the demand function intersects the price axis) the second term blows up (approaches infinity), because we’re coming close to dividing by zero, and therefore
the elasticity approaches minus-infinity (since the slope is negative). On the other hand, as we approach the zero point on the price axis, the second term is zero, and so is elasticity. It turns out that along a linear demand curve, elasticities take on all possible values from 0 to minus-infinity.

To verify this, we can compute the elasticity at \( p_A = 3.00 \). Quantity demanded is \( 16 - (2.1 \times 3.00) + 6 + (0.001 \times 40000) = 55.7 \), so the elasticity at this point is

\[
\eta = \frac{-2.1 \times 3.00}{55.7} = -0.11311
\]

which of course is different from our previous answer.

4. Consider a 2-region economy producing a single good, \( h \). Inter-regional transportation is possible at a cost of 2 per unit shipped. In region \( A \) the demand for \( h \) is

\[
h_A^D = 43 - 2p_A
\]

and the supply function is given by

\[
h_A^S = 16 + 3p_A
\]

where \( p_A \) is the own-price of \( h \) in region \( A \).

(a) What is the autarkic equilibrium price in region \( A \)?
(b) If in autarky we observe \( p_B = 8.15 \), what is the direction of inter-regional trade?
(c) Assuming that trade takes place, what can you say about the equilibrium levels of prices, relative to the autarkic levels?

**Answer Sketch:** For this 2-region system:

(a) To find the autarkic equilibrium set demand equal to supply and solve. We have

\[
43 - 2p_A = 16 + 3p_A \\
43 - 16 = 3p_A + 2p_A \\
27 = 5p_A \\
p_A = \frac{27}{5} = 5.4
\]

(b) Since we know that the autarkic price in region \( B \) is higher than 5.4, we conclude that trade will flow from \( A \) to \( B \). (\( A \) is the exporter, \( B \) the importer).

(c) The equilibrium price in the exporting region will be greater than 5.4 and the equilibrium price in the importing region will be lower than 8.15; and the two will differ by exactly 2 (the transport costs). It is important to realize that without knowing more about the demand and supply curves in region \( B \) we can’t say more than that (so there’s no point in trying to compute excess demands and supplies, for example).

5. Consider a small 1-class monocentric city, with all assumptions as in class. Suppose the (city) transportation cost decreases. Analyze the impact on the city size (extent). You should assume that only the transportation cost changes.

**Answer Sketch:** The extent of the city is determined by the bid-rent of city dwellers as compared to the agricultural rent, which is fixed at \( R_A \). So our first step is to determine what happens to city bid-rent.
The picture below shows the standard setup, with $Y(s)$ as the initial transportation cost. After the decrease in transportation cost, the disposable income line pivots outwards as to $Y'(s)$, since $Y(s) = M - ks$ and $Y'(s) = M - k's$, with $k' < k$, and $Y(o) = Y'(0) = M$ and the slope of $Y'(s)$ is lower. This tells us that land rents at the CBD are unaffected by the transportation cost change.

Away from the CBD, fix distance at $s_1$. The initial bid-rent at $s_1$ is the slope of budget line A, which goes through $Y(s_1)$ and is tangent to the common spatial-equilibrium indifference curve $I^*$. After the transportation cost decrease, disposable income at $s_1$ is $Y'(s_1)$, and the bid rent at $s_1$ is the slope of line B, tangent to the same $I^*$. Line B is clearly steeper than A, which means that the impact of the transportation change is that bid-rents increase at every distance except at the CBD. The figure below shows the upshot: $R(s)$ is the initial bid-rent, while $R'(s)$ is the one that would prevail after transportation costs decrease. Given the agricultural land rent $R_A$ we see that the impact on the city extent is that the city expands.

6. The Director in a certain School of Architecture lives in a 2-good world. The two goods are (a) trips to Hong Kong and (b) Beluga caviare. Suppose that the price of a trip is $2500, while caviare costs $145 per jar. If we force her to curtail her trips by one per year, how many jars of caviare must we give her in order to keep her on the same indifference curve?

**Answer Sketch:** The important thing to understand here is that we are using the slope of the budget constraint to make inferences about the slope of the indifference curve (given that we believe that our individual is in equilibrium, where the two are tangent). Let $T$ and $J$ be trips and jars of caviare, and $p_T$ and $p_J$ their prices. In equilibrium we have (since in equilibrium the slopes of the indifference curve and the budget constraint are equal)

$$\frac{\Delta T}{\Delta J} = -\frac{p_J}{p_T}$$
(this corresponds to plotting Trips on the vertical axis and Caviare on the horizontal one). We know that $\Delta T = -1$ so we have

$$\Delta T = -\frac{p_J}{p_T} \Delta J$$
$$\Delta J = -\Delta T \frac{p_T}{p_J}$$

then with out data, $p_T/p_J = 2500/145 = 17.241$ so $\Delta J = 17.241$ (rounding). We would need to give her 17.2 jars. (An early draft of this problem tried to heed the sartorial habits of the architecture faculty, so the second good was Black Suits. But the numbers ended up looking weird).

7. Consider two groups (the X’s and the Y’s) in a small monocentric city (as in class). The bid rents of the two groups are

$$R_X(s) = 30 - 2s$$
$$R_Y(s) = 20 - s$$

respectively, and where, as usual, $s$ measures distance from the CBD. Who lives where in this city?

**Answer Sketch:** First, observe that both bid rents are straight lines. The non-$s$ terms (ie 30 and 20) are the intercepts, ie the bids for land when $s = 0$. The X’s have a higher intercept than the Y’s. So the X’s bid higher for land at $s = 0$, ie for land near the CBD, and they get it. Note that this isn’t quite sufficient, though: if the slope of $R_Y(s)$ was *less* than that of $R_X(s)$ then the Y’s would *never* outbid the X’s : they’d be “crowded out” of the landscape, and would presumably end up as farmers. But in this case the slope of $R_X(s)$ is $-2$ and that of $R_Y(s)$ is $-1$: this means that $R_X(s)$ starts off higher, but decreases faster than $R_Y(s)$, ie that at some point the two will intersect. See picture below.

So where is the “integrated district”, where the two groups live side by side? To find the location of this district, just find the point at which the bid-rents are equal, ie solve $30 - 2s = 20 - s$ for $s$. Solving, it turns out that $s^* = 10$. 

![Rent diagram](image-url)