1. Consider a 1-class small monocentric open city. Suppose that for some reason the spatial equilibrium utility level for all city dwellers goes down (and that nothing else changes). What is the impact on (a) the equilibrium rent pattern in the city; and (b) the extent of the city?

Answer Sketch: We will analyze this using our conventional picture:

![Diagram of disposable-income line and indifference curves]

The disposable-income line is $Y(s) = M - ks$, and the original spatial-equilibrium utility level is $I_1$, while the new (lower) level is $I_2$. Consider the individual living at $s_1$. Her original bid-rent for land at $s_1$ must go through $Y(s_1)$ and must be tangent to $I_1$; this is line $A$. After the utility change, her bid-rent must still go through $Y(s_1)$ — neither the transport cost nor the price of $z$ has changed — but now must be tangent to $I_2$. This generates line $B$. Clearly $B$ is steeper than $A$; and since the slope of a budget (bid-rent) line is (minus) the price of land, we see that her bid for land at $s_1$ — which, with just one class, is also the equilibrium price of land at $s_1$ — must have gone up.

It should be obvious that precisely the same thing will hold true at the CBD; and also at any other location away from the CBD. Therefore we conclude that land prices inside the city go up at all locations.

Could we have reached this conclusion without the diagram? Sure: in the new situation, everyone must be worse off (on a lower indifference curve). Income doesn’t change, nor does the transport cost or the price of the composite commodity. The only way you could be worse off is if land prices everywhere increase; and that is what happens here.
As to the extent of the city, it is easy to see that if bids for land go up everywhere and the price of agricultural land doesn’t change (by assumption) then the city must expand: some land where the farmers previously outbid the city-dwellers will now change hands as the city folk bid more.

2. Use indifference curves and budget lines to illustrate the possibility that a demand function can slope upwards (in own-price). Be sure to label everything carefully, and include a brief explanation of your diagram.

**Answer Sketch:** What do we need to show? This can be put in two equivalent ways. We want to show that, for our chosen good, an own-price increase leads to an increase in the quantity consumed; or else that a price decrease leads to a decrease in quantity consumed. In other words, that own-price changes and consumption quantities move in the same direction, unlike the standard case where the move in opposite directions. And we want to do this with a “legal” diagram, in particular one where the indifference curves obey the usual rules.

See the figure below: at some original prices we have the budget constraint \( A \), and the quantity of \( x_1 \) consumes is \( x_{11} \). Now suppose the price of good 1 increases. Then the budget constraint pivots inwards, say to \( B \). We now need to draw a tangency where consumption of \( x_1 \) increases. As drawn in the picture, we get a new consumption quantity \( x_{12} > x_{11} \). So a price increase results in a demand increase.

Note a couple of things: drawing the second indifference curve isn’t completely straightforward, since you have to do so in such a way that the two indifference curves don’t cross. This means that the second curve isn’t going to have the same “shape” as the first. The fact that this is a bit difficult reflects the fact that this kind of behavior is unusual. Second, note that the construction doesn’t require that one of the goods be addictive. It can be, but note that the graphical difficulty in drawing the second indifference curve remains. Finally, you should be able to convince yourself that while one good can have an upward-sloping demand curve, it’s not possible that both goods do. (The generalization of this is that not all goods can have upward-sloping demands). Finally, remember the caveat also mentioned in class: while it is certainly theoretically possible for demand functions to slope upwards, in practice they almost never do. You need a very strong argument to base any policy prescriptions on the case of upward-sloping demand.

3. Consider a 2-region economy, and a single good \( x \). The demand and supply functions in Region 1 are:

\[
x_{1D} = 56 - 2p_1 \\
x_{1S} = 12 + 4p_1
\]
where \( p_1 \) is the own-price. In Region 2 we have

\[
\begin{align*}
    x_2^D &= 28 - p_2 \\
    x_2^S &= 8 + 3p_2
\end{align*}
\]

Based on this information:

(a) Find the autarkic equilibrium price in Region 1.
(b) Find the autarkic equilibrium price in Region 2.
(c) Now assume that inter-regional trade opens up, and that goods can be transported between the two regions at a transportation cost of 2 per unit shipped. If trade takes place at all, in which direction will it flow? (That is, which region will be the importing region and which will be the exporting region)?
(d) Write down an expression for the excess supply function for the exporting region.
(e) Write down an expression for the excess demand function for the importing region.
(f) In equilibrium, what must be true of the two prices \( p_1 \) and \( p_2 \)?
(g) Using the information in the previous three parts, find the equilibrium prices after trade opens up.

**Answer Sketch:** For this region:

(a) To find the autarkic equilibrium price in Region 1, set demand equal to supply and solve:

\[
\begin{align*}
    x_1^D &= x_1^S \\
    56 - 2p_1 &= 12 + 4p_1
\end{align*}
\]

Collecting terms:

\[
\begin{align*}
    56 - 12 &= 4p_1 + 2p_1 \\
    44 &= 6p_1 \\
    p_1^* &= 44/6. = 7.3333
\end{align*}
\]

(b) Same thing for Region 2:

\[
\begin{align*}
    28 - p_2 &= 8 + 3p_2 \\
    28 - 8 &= 3p_2 + p_2 \\
    20 &= 4p_2 \\
    p_2^* &= 20/4 = 5.
\end{align*}
\]

(c) Trade will flow from the low-price region to the high-price region, so in our case this means that the exporting region is Region 2 and the importing region is Region 1.

(d) The excess supply function for the exporting region is the excess supply function for Region 2; and excess supply is just supply minus demand, so:

\[
\begin{align*}
    ES_2 &= x_2^S - x_2^D \\
    &= 8 + 3p_2 - (28 - p_2) \\
    &= 8 + 3p_2 - 28 + p_2 \\
    &= 4p_2 - 20
\end{align*}
\]
(e) The excess demand function for the importing region is the excess demand function for Region 1; and excess demand is demand minus supply, so:

\[ E_D_1 = 56 - 2p_1 - (12 + 4p_1) \]
\[ = 56 - 2p_1 - 12 - 4p_1 \]
\[ = 44 - 6p_1 \]

(f) In equilibrium the prices must differ by exactly the transportation cost. Specifically, the price in the importing region (the higher price) minus the price in the exporting region must equal the transportation cost. In our case this means \( p_1 - p_2 = 2 \) where the transportation cost is 2. Or, more usefully, \( p_1 = 2 + p_2 \)

(g) To find the inter-regional equilibrium, we need to solve simultaneously the conditions

\[ E_S_2 = E_D_1 \]
\[ p_1 = 2 + p_2 \]

Plugging in, this means we need to solve

\[ 4p_2 - 20 = 44 - 6p_1 \]
\[ p_1 = 2 + p_2 \]

Use the second equation to substitute for \( p_1 \) in the first, giving

\[ 4p_2 - 20 = 44 - 6(2 + p_2) \]
\[ 4p_2 - 20 = 44 - 12 - 6p_2 \]

Now collect up terms, as usual:

\[ 4p_2 + 6p_2 = 44 - 12 + 20 \]
\[ 10p_2 = 52 \]
\[ p_2 = 5.2 \]

which means that

\[ p_1 = 2 + p_2 = 7.2 \]

Note that not much has changed: you can plug in and see that the amount of \( x \) that is shipped inter-regionally is 0.8 units.

4. Suppose that production of a good \( x \) is characterized by Leontief (L-shaped) isoquants. What will be the impact of changes in relative prices on the cost-minimizing factor demands (ie, input utilization) for a given output level?

**Answer Sketch:** See the figure, where we show a target L-shaped isoquant. Line \( A \) is an isocost line. The cost-minimizing factor demands are found by moving \( A \) inwards in parallel until it just touches the isoquant. The result is line \( AA \) parallel to \( A \), and the tangency occurs at the kink in the isoquant.
What happens at different factor prices? The slope of $A$ changes, and we get a new iso-cost line $B$. Proceeding as before, we find that the cost-minimizing factor demands are again at the kink in the isoquant (line $BB$). A bit of experimentation will show you that at any relative prices (as long as both prices are positive numbers) we will always get this result. In other words, the cost-minimizing factor demands are independent of relative prices.

Is this surprising? The Leontief isoquants represent production under fixed factor proportions, ie without the possibility of substitution. So as prices change, there’s nothing you can do to mitigate the impact, as long as you’re constrained to continue producing on the same isoquant.

The fact that you can work out the inputs for any output level without knowing relative prices makes this model a popular one. But you should realize that it is thought to be quite unrealistic in the real world: the fact is that some form of input substitution is often possible; and this is what the Leontief model denies.

5. Consider production using a form of the famous Cobb-Douglas production function

$$x = 23(z_1^{0.6})(z_2^{0.5})$$

where the $z$’s are the inputs. Suppose that we currently produce using the input bundle $(z_1, z_2) = (5, 4)$. What can you say about returns to scale if you were to triple the scale of operations?

**Answer Sketch**: The question is: if we triple all inputs, does output triple (CRTS), does it go up by less than a factor of three (DRTS) or by more than a factor of three (IRTS)? This is easy.

First, what is the output at the given input bundle? It is

$$x = 23(5^{0.6})(4^{0.5})$$

$$= 120.82$$

Next, what if we triple all inputs? The new input bundle is $(15,12)$, so output is

$$x' = 23(15^{0.6})(12^{0.5})$$

$$= 404.55$$

The proportionate increase in output is thus

$$404.55/120.82 = 3.348$$
We tripled all inputs, and output has gone up by a factor greater than three. This is IRTS.

In fact you can do this without the tripling. Let all inputs increase by a factor of \( \lambda > 1 \). Then the new output is

\[
x' = 23(\lambda z_1)^{0.6}(\lambda z_2)^{0.5} \\
= 23\lambda^{1.1} z_1^{0.6} z_2^{0.5} \\
= \lambda^{1.1} x
\]

(where the third line relies on properties of exponents). So the proportionate increase in output is \( \lambda^{1.1} x/x = \lambda^{1.1} \). This is clearly going to be greater than \( \lambda \), so we have IRTS. This generalizes in the obvious way: write the Cobb-Douglas production function in \( n \) inputs as

\[
x = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}
\]

It is easy to show that we will have CRTS if \( a_1 + a_2 + \cdots + a_n = 1 \); we will have DRTS if \( a_1 + a_2 + \cdots + a_n < 1 \); and we have IRTS if \( a_1 + a_2 + \cdots + a_n > 1 \).

6. Consider an individual whose demand for a good or service \( x \) is given by

\[
x^* = 56 - 2p_1 + p_2 + 0.0001M
\]

where \( p_1 \) is the price of \( x \); \( p_2 \) is the price of some other good; and \( M \) is income. Suppose we observe \( p_1 = 2.00 \); \( p_2 = 3.00 \) and \( M = 50,000 \).

(a) Compute the own-price elasticity of demand for \( x \).

(b) Compute the income elasticity of demand for \( x \).

(Hint: be careful with the number of zeros after the decimal point in the coefficient of \( M \) in the demand equation).

**Answer Sketch:** You can do this question in two ways: using calculus, or by finite changes. In the present case, because the demand function is linear in everything, the two methods will give the same results; but this is not generally true. If you are going to use the finite change method, you need to generate a second point by considering a small change from the first one.

Now to work. First of all, we need to find the demand at the given prices and income. This is simple: we have

\[
x^* = 56 - 2p_1 + p_2 + 0.0001M \\
= 56 - (2 \times 2.00) + 3.00 + (0.0001 \times 50000) \\
= 56 - 4 + 3 + 5 \\
= 60
\]

(a) For the own price elasticity: the formula is for the own-price elasticity is

\[
E = \frac{\Delta x/x^0}{\Delta p/p_1^0} = \frac{\Delta x}{x^0} / \frac{\Delta p}{p_1^0} = \frac{\Delta x}{p_1^0} \frac{p_1^0}{x^0} \Delta p
\]
where the superscript zeros indicate the base position. To find $\Delta x$ we need one other demand point, and we generate this by looking at a small price change. A 1¢ change is probably small, so let’s see what happens at a price of $x$ of 2.01. We have

$$x^* = 56 - 2p_1 + p_2 + 0.0001M$$

$$= 56 - (2 \times 2.01) + 3.00 + (0.0001 \times 50000)$$

$$= 59.98$$

(where $p_1'$ is the new price of $x_1$). Note that the only thing that has changed in the calculation is the price of $x$. Then $\Delta p = 0.01 = 2.01 - 2.00$ and $\Delta x = 59.98 - 60 = -0.02$. (Remember that you must do the subtractions in the same order: here I’ve done “new” – “old”, ie $p_1' - p_1$. If I’d done if as “old” – “new” the signs would change, but the numbers wouldn’t). Then

$$E = \frac{\Delta x}{\Delta p} \frac{p^0}{x^0}$$

$$= \frac{-0.02}{2.00} \frac{2.00}{60}$$

$$= -0.0667$$

One reason for brushing up on your calculus is that it makes the whole thing much easier. The definition in this case is

$$E = \frac{dx^*}{dp} \frac{p^0}{x^0}$$

with the derivative evaluated at the original or base point. In this case, by inspection, $dx^*/dp = -2$, so

$$E = -2 \frac{2.00}{60} = -0.0667$$

as before. But note that the fact that they both come out the same way is because everything’s linear. If the demand had been non-linear in own-price there would have been a small difference.

(b) The definition of the income elasticity of demand is

$$\frac{\% \text{ change in quantity demanded}}{\% \text{ change in income}}$$

which comes down to

$$E = \frac{\Delta x}{\Delta M} \frac{x^0}{M^0} = \frac{\Delta x}{\Delta M} \frac{M^0}{x^0}$$

We pick a small change in income, say $1 (this is small in the context of 50,000: it wouldn’t necessarily be small in the context of a price of 2.00), and see what happens at an income of say 49,999. We have

$$x^* = 56 - 2p_1 + p_2 + 0.0001M'$$

$$= 56 - (2 \times 2.00) + 3.00 + (0.0001 \times 49999)$$

$$= 59.9999$$

Then, doing the computations as “old”–“new”, for variety we have $\Delta x = 60 - 59.9999 = 0.0001$; $\Delta M = 50000 - 49999 = 1$ so

$$E = \frac{0.0001 \times 50000}{1} \frac{1}{60}$$

$$= 0.00833$$
Using calculus it’s faster since we want to compute

\[ E = \frac{dx^* M^0}{dM x^0} \]

and the derivative \( dx^*/dM \) is 0.0001 by inspection. Then we get

\[ E = 0.001 \frac{50000}{60} \]

obviously the same as before; but once again note that this is so just because the demand function is linear in income.

7. Consider a utility-maximizing price-taking individual in a 2-good world. Explain geometrically (assuming that you possess all the necessary data) how to find the minimum income needed to allow him or her to achieve a given utility level \( u^* \).

**Answer Sketch:** Note that this question asked you to show how to do it, not just to show the final answer. So consider the picture below.

![Diagram of indifference curve and budget line](image)

The indifference curve \( u^* \) is our target curve. Now suppose we had a budget of $M. At fixed prices \( p_1 \) and \( p_2 \) for goods \( x_1 \) and \( x_2 \), feasible quantities satisfy \( p_1 x_1 + p_2 x_2 = M \). This is of course a budget line, say \( A \) in the picture. We now need to find the minimum budget that, at fixed prices, will allow you to reach the target indifference curve. So we move line \( A \) in towards the origin as far as possible in parallel, until it just touches the target indifference curve. The result is \( AA \), and the least-income quantities are \( x_{11} \) and \( x_{12} \). Given these (or any two points on \( AA \)) we can calculate the least income needed as \( M^* = p_1 x_{11} + p_2 x_{21} \).

Sound familiar? It should. Except for the labelling, this is precisely the same construction we used to derive the (total) cost function for a cost-minimizing public or private producer. In the individual-behavior context, the construction gives what advanced texts refer to as the Expenditure Function: this tells you the least income you need, at fixed prices, to achieve the utility level \( u^* \): it is usually written \( E^*(p, u^*) \), where \( p \) is an abbreviation for the whole list (vector) of consumer prices. It turns out that the expenditure function is very useful in a variety of public finance contexts, for example, in computing compensation for market-based policies. One difficulty, though, is that it depends on knowing the target utility level, and as we’ve argued, this is hardly straightforward. Most uses of the expenditure function are therefore theoretical, though it turns out that the derivatives of the expenditure function are both easy to compute and observable.