Answers are due in class in one week.

1. Consider the problem of optimizing $f(x_1, x_2, x_3)$. Write down all the second-order conditions which must hold at an extremal point $x^*$ if that point is to be

(a) a maximum and 
(b) a minimum.

2. (continuation) Suppose now that we optimize the function $f$ of the previous question subject to the constraint $g(x_1, x_3) = 0$. What now are the second-order conditions for (a) a maximum and (b) a minimum?

3. (Miscellaneous facts about homogeneous functions) A function $f(x_1, x_2 \ldots x_n)$ is said to be homogeneous of degree $t$ in $(x_1, x_2 \ldots x_n)$ if and only if, for any $k > 0$

$$f(kx_1, kx_2 \ldots kx_n) \equiv k^t f(x_1, x_2 \ldots x_n)$$

The case $t = 1$ is often called linear homogeneity. The following three results are widely used in economics; proofs can be found in, eg, Silberberg, pp. 90, 98.

**Euler’s Theorem** Suppose $f$ is homogeneous of degree $t$. Then

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots + \frac{\partial f}{\partial x_n} x_n \equiv t f(x_1, x_2 \ldots x_n)$$

**Theorem 2** Suppose $f$ is homogeneous of degree $t$. Then the function defined by $g = \frac{\partial f}{\partial x_1}$ is homogeneous of degree $t - 1$.

**Theorem 3** Suppose $y = f(x_1, x_2)$ is homogeneous of degree $t = 1$. Then

$$\frac{y}{x_1} = f(1, \frac{x_2}{x_1}) = g(\frac{x_2}{x_1})$$

that is, only the ratios matter. (Proof: in Theorem 1, take $k = 1/x_1$). Note that this will extend to $f$ depending on any number of the $x$’s.

With these results in mind:

(a) verify that $y = x_1^{(0.4)} x_2^{(0.6)}$ is homogeneous of degree 1 in $x_1$ and $x_2$, and that Euler’s Theorem and Theorems 2 and 3 hold.

(b) what about $y = x_1^{(0.4)} x_2^{(0.4)}$?

4. Consider the problem

Max $\quad f = x_0 + a_1 \ln x_1$

s.t. $\quad a_2 = x_0 + a_3 x_1$

where all the $\alpha$’s are strictly positive.
(a) Sketch the objective function and the constraint qualitatively (ie show their general shapes).
(b) Set up the Lagrangian and obtain the FOCs.
(c) Solve for $x_0^*$ and $x_1^*$.
(d) Obtain the SOC and verify that your solution is indeed a maximum.
(e) Obtain the indirect objective function $f^*(\alpha)$
(f) From your calculation of $f^*$, compute $\partial f^*/\partial \alpha_i$ for $i = 1, \ldots, 3$.
(g) Use the Envelope Theorem to obtain $\partial f^*/\partial \alpha_i$ for $i = 1, \ldots, 3$ and verify that your answers agree with those obtain in the previous part.

5. (continuation) Consider the problem

\[
\begin{align*}
\text{Min} & \quad g = x_0 + \alpha_3 x_1 \\
\text{s.t.} & \quad x_0 + a_1 \ln x_1 = (a_2 - a_1) + a_1 \ln(a_1/a_3)
\end{align*}
\]

where all the $\alpha$’s are as in the previous problem and are (still) strictly positive.

(a) What is the relation of this problem and the previous one? Hint: think of the constraint as $x_0 + \alpha_3 x_1 = A$. Does $A$ look familiar? Sketch the objective function and the constraint qualitatively. Can you say without further work what the solution of this problem is?
(b) Solve the problem explicitly.
(c) Check that your solution is a minimum.
(d) Obtain the indirect objective function.
(e) Obtain the derivatives of the indirect objective function with respect to the parameters. Can you determine the signs of these derivatives?

6. Consider the problem: choose $x$ and $y$ to

\[
\begin{align*}
\text{Minimize} & \quad M = \alpha x + \beta y \\
\text{subject to} & \quad xy = U
\end{align*}
\]

with $\alpha, \beta > 0$.

Suppose you know that the solutions $x^*(\alpha, \beta, U)$ and $y^*(\alpha, \beta, U)$ are always positive.

(a) solve the problem directly, using the method of Lagrange multipliers. Find the choice functions $x^*$ and $y^*$ explicitly; use them to find the indirect objective function $M^*$ explicitly, and then differentiate $M^*$ with respect to $\alpha$, $\beta$, and $U$ and determine the signs of the derivatives.
(b) Use the Envelope Theorem to write down without any computation, expressions for $\partial M^*/\partial \alpha$, $\partial M^*/\partial \beta$ and $\partial M^*/\partial U$. Verify that these agree with what you found in part (a); and note how quick it was this way.