1 Introduction

This note states the second-order sufficient conditions for optimization, in a form useful for comparative-statics derivations. Reference: E. Silberberg, The Structure of Economics, Chapter 6 and Review of Chapter 5 of the third edition (Chapter 9, and review of Chapter 5 of the first edition).

Let \( x = (x_1, x_2, \ldots, x_N) \) be an \( N \)-vector of choice variables, and let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M) \) be an \( M \)-vector of parameters. Let the optimand be \( f(x; \alpha) \), and let there be \( R \) constraint functions \( g_r(x; \alpha) \), \( r = 1, 2, \ldots, R \). We assume that both \( f \) and the \( g_r \) are continuous and have continuous first and second partial derivatives.

2 Unconstrained optimization

2.1 Problem Setup

In this case \( R = 0 \) and our problem is to choose \( x \) to optimize \( \mathcal{L} = f \). (We write the problem in this way to emphasize the formal similarity between the results for constrained and unconstrained optimization).

A necessary condition for an optimum is that \( \partial \mathcal{L} / \partial x_i = \partial f / \partial x_i = 0 \) for all \( i \). These are the first-order conditions (FOCs); a point \( x^* \) satisfying them is said to be
a *stationary point*. The same FOCs hold for both a maximum and a minimum. We want to decide, once we have a stationary point, which kind (max or min) it is.

### 2.2 The Hessian

The *Hessian* matrix $H$ of $L$ is the matrix of second partial derivatives of $L$:

$$
H = \begin{bmatrix}
\frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_N} \\
\frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 L}{\partial x_N \partial x_1} & \frac{\partial^2 L}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_N^2}
\end{bmatrix}
$$

### 2.3 Principal minors

Let $H$ be an $(N \times N)$ matrix. Consider forming a smaller matrix by striking out $N - k$ columns of $H$, and striking out also the same-numbered rows. That is, if you strike out (for example) columns 1, 4, and 12 of $H$, then you must strike out rows 1, 4, and 12 also. The result is a matrix of size $N - (N - k) = k$ rows and columns, ie a $(k \times k)$ matrix. The determinant of this $(k \times k)$ matrix is a **principal minor of order** $k$ of the original matrix $H$. Note that:

In a 0-constraint problem, a principal minor of order $k$ is the determinant of a $k \times k$ matrix.

Also, note that

- for any $k$ there may be more than one principal minor of order $k$ (since you may be able to delete $N - k$ rows and columns in different ways);
- the principal minors of order 1 are the elements of $H$ on the main diagonal, $\frac{\partial^2 L}{\partial x_i^2}$;
- the determinant of the entire matrix is a principal minor of order $N$.

### 2.4 Second-order conditions for unconstrained optimization

At a stationary point $x^*$ satisfying the FOCs:
**Unconstrained maximization** If, for all \( k = 1, 2, \ldots, N \), the principal minors of order \( k \) have sign \((-1)^k\) then \( f \) has a maximum at \( x^* \). (Or: the principal minors of the Hessian alternate in sign, beginning with the negative).

**Unconstrained minimization** If, for all \( k = 1, 2, \ldots, N \), the principal minors of order \( k \) are positive then \( f \) has a minimum at \( x^* \).

### 3 Equality-constrained optimization

#### 3.1 Problem setup - 1 equality constraint

Consider the problem of finding an extremum of \( f(x; \alpha) \) subject to the one equality constraint \( g^1(x; \alpha) = 0 \). Form the Lagrangian

\[
\mathcal{L} = f + \lambda_1 g^1
\]

The FOC’s are that \( \partial \mathcal{L} / \partial x_i = 0 \) for all \( i = 1, 2, \ldots, N \); and additionally that \( \partial \mathcal{L} / \partial \lambda_1 = 0 \). Note the surface similarity to the previous problem: we are interested in extremal points of \( \mathcal{L} \).

#### 3.2 The bordered Hessian - 1 equality constraint

We define the Hessian matrix exactly as we did before, except that we take the partial derivatives with respect to the augmented set of choice variables, the \( x \)'s and the Lagrange multiplier \( \lambda_1 \). The Hessian for the 1-constraint problem is therefore a square \((N + 1) \times (N + 1)\) matrix called the *bordered Hessian*:

\[
H = \begin{bmatrix}
\frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_N} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\
\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \ldots & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_N} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 \mathcal{L}}{\partial x_N \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_N \partial x_2} & \ldots & \frac{\partial^2 \mathcal{L}}{\partial x_N^2} & \frac{\partial^2 \mathcal{L}}{\partial x_N \partial \lambda} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \ldots & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_N} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2}
\end{bmatrix}
\]

Why “bordered” Hessian? Look at \( \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \) — the entry at the end of the
first row. This is

\[
\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_1} = \frac{\partial}{\partial x_1} \left( \frac{\partial \mathcal{L}}{\partial \lambda_1} \right)
= \frac{\partial}{\partial x_1} g^1
= \frac{\partial g^1}{\partial x_1}
\]

The same holds as we proceed down the last column or across the last row. And when we reach the last element in the last row we compute

\[
\frac{\partial \mathcal{L}}{\partial \lambda_1} = \frac{\partial g^1}{\partial \lambda_1} = 0
\]

In other words, the bordered Hessian is just the original Hessian matrix “bordered” on the right and below by the first-order partial derivatives of the constraint $g^1$ with respect to the set of augmented choice variables:

\[
H = \begin{bmatrix}
\frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_N} & \frac{\partial g^1}{\partial x_1} \\
\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_N} & \frac{\partial g^1}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 \mathcal{L}}{\partial x_N \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_N \partial x_N} & \frac{\partial g^1}{\partial x_N} \\
\frac{\partial g^1}{\partial x_1} & \frac{\partial g^1}{\partial x_2} & \cdots & \frac{\partial g^1}{\partial x_N} & 0
\end{bmatrix}
\]

### 3.3 Border-preserving principal minors

As before, we shall delete $N - k$ rows and corresponding columns from the (bordered) Hessian. But this time, there is an added consideration: we may select for deletion only rows and columns from among the first $N$, that is, we do not delete the border (last row and column) itself. (Of course, deleting rows and columns drops elements of the border; but we may not delete the border as a whole). The determinant of the result is a border-preserving principal minor of order $k$. Since the bordered Hessian is an $(N + 1) \times (N + 1)$ matrix, and we delete $N - k$ rows and columns, we find that since $(N + 1) - (N - k) = k + 1$:

In a 1-constraint problem, a border-preserving principal minor of order $k$ is the determinant of $(k + 1) \times (k + 1)$ square matrix.
3.4 Second-order conditions: one equality constraint

At a stationary point \((x^*, \lambda_1^*)\) satisfying the FOCs:

**Constrained maximization** If for each \(k \geq 2\) the border-preserving principal minors of order \(k\) have sign \((-1)^k\), then \(f\) has a maximum at \(x^*\) subject to \(g^1 = 0\).

**Constrained minimization** If all the border-preserving principal minors of order \(k \geq 2\) have sign \((-1)^k\), i.e., are all negative, then \(f\) has a minimum at \(x^*\) subject to \(g^1 = 0\).

Note that when \(R = 1\) the minors we are interested in begin with order \(k = 2\), while with no constraints \((R = 0)\) they begin with \(k = 1\). With a view towards generalization, note that both setups can be written as follows:

**Maximization** If for each \(k > R\) the border-preserving principal minors of order \(k\) have sign \((-1)^k\), then \(f\) has a maximum at \(x^*\) subject to \(g^1 = 0\).

**Minimization** If all the border-preserving principal minors of order \(k > R\) have sign \((-1)^k\), i.e., are all negative, then \(f\) has a minimum at \(x^*\) subject to \(g^1 = 0\).

3.5 Two equality constraints

Suppose you have \(R = 2\) independent equality constraints, \(g^1 = 0\) and \(g^2 = 0\); and you wish to optimize \(f\) subject to both. Form the Lagrangian, introducing one Lagrange multiplier for each constraint:

\[ \mathcal{L} = f + \lambda_1 g^1 + \lambda_2 g^2 \]

The FOCs require that \(\partial \mathcal{L}/\partial x_i = 0, i = 1, 2, \ldots, N\), and \(\partial \mathcal{L}/\partial \lambda_j = 0, j = 1, 2\).

To form the bordered Hessian we proceed exactly as before, except that now the border has two rows and columns: the second-to-last column is the vector of the derivatives of the first constraint \(g_1\) with respect to \((x, \lambda_1, \lambda_2)\) and the last column of the bordered Hessian is the vector of the derivatives of the second constraint \(g_2\). The bordered Hessian is therefore a square matrix of size \(N + 2\).
Border-preserving principal minors of order $k$ are formed exactly as before: strike out $N - k$ rows and corresponding columns of the bordered Hessian, taking these only from the first $N$ rows and columns. Therefore, since $(N+2) - (N-k) = k + 2:

\begin{align*}
H &= \begin{bmatrix}
\frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_N} & \frac{\partial g^1}{\partial x_1} & \frac{\partial g^2}{\partial x_1} \\
\frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_N} & \frac{\partial g^1}{\partial x_2} & \frac{\partial g^2}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\partial^2 L}{\partial x_N \partial x_1} & \frac{\partial^2 L}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_N^2} & \frac{\partial g^1}{\partial x_N} & \frac{\partial g^2}{\partial x_N} \\
\frac{\partial g^1}{\partial x_1} & \frac{\partial g^1}{\partial x_2} & \cdots & \frac{\partial g^1}{\partial x_N} & 0 & 0 \\
\frac{\partial g^2}{\partial x_1} & \frac{\partial g^2}{\partial x_2} & \cdots & \frac{\partial g^2}{\partial x_N} & 0 & 0
\end{bmatrix}
\end{align*}

In a 2 constraint problem, a border-preserving principal minor of order $k$ is the determinant of a $(k + 2) \times (k + 2)$ square matrix.

The second-order conditions are generalizations of the conditions we have already obtained, where we write $g = 0$ for $(g_1 = 0, g_2 = 0)$. At a stationary point $(x^*, \lambda_1^*, \lambda_2^*)$ satisfying the FOCs:

**Maximization** If for each $k > R$ the border-preserving principal minors of order $k$ have sign $(-1)^k$, then $f$ has a maximum at $x^*$ subject to $g^1 = 0$ and $g^2 = 0$.

**Minimization** If all the border-preserving principal minors of order $k > R$ have sign $(-1)^2$, ie are all positive, then $f$ has a minimum at $x^*$ subject to $g^1 = 0$ and $g^2 = 0$.

Of course, note that with 2 constraints we look at border-preserving principal minors of order $> 2$, ie 3 and above; while with 1 constraint we examine order $> 1$, ie 2 and above.

4 Second-order conditions in general

Suppose we are interested in optimizing $f(x; \alpha)$ subject to $R (\geq 0)$ independent equality constraints. Form the Lagrangian

$$L = f(x; \alpha) + \lambda \cdot g(x; \alpha)$$
where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_R) \) and \( g(x; \alpha) = (g^1(x; \alpha), g^2(x; \alpha), \ldots, g^R(x; \alpha)) \). The FOCs are that \( \frac{\partial L}{\partial x_i} = 0, i = 1, 2, \ldots, N \) and \( \frac{\partial L}{\partial \lambda_j} = 0, j = 1, 2, \ldots, R \).

Form the square \( N + R \) bordered Hessian:

\[
H = \begin{bmatrix}
\frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_N} & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^R}{\partial x_1} \\
\frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_N} & \frac{\partial g^1}{\partial x_2} & \cdots & \frac{\partial g^R}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 L}{\partial x_N \partial x_1} & \frac{\partial^2 L}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_N \partial x_N} & \frac{\partial g^1}{\partial x_N} & \cdots & \frac{\partial g^R}{\partial x_N} \\
\frac{\partial g^1}{\partial x_1} & \frac{\partial g^1}{\partial x_2} & \cdots & \frac{\partial g^1}{\partial x_N} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial g^R}{\partial x_1} & \frac{\partial g^R}{\partial x_2} & \cdots & \frac{\partial g^R}{\partial x_N} & 0 & \cdots & 0
\end{bmatrix}
\]

To form a border-preserving principal minor of order \( k \), we still delete \( N - k \) rows and columns from the first \( N \) rows and columns (i.e., those not containing any part of the border). Therefore since \( (N + R) - (N - k) = k + R \):

In an \( R \)-constraint problem, a border-preserving principal minor of order \( k \) is the determinant of a \( (k + R) \times (k + R) \) square matrix.

Generalizing the previous cases, the second-order conditions involve the signs of all the minors of orders \( k > R \). We now state those conditions in two equivalent forms.

### 4.1 Second-order conditions: formulation 1

This case is a direct generalization of previous results. At a stationary point \((x^*, \lambda^*)\) satisfying the FOCs:

**Maximization** If for each \( k > R \) the border-preserving principal minors of order \( k \) have sign \((-1)^k\), then \( f \) has a maximum at \( x^* \) subject to the \( R \) constraints \( g = 0 \).

**Minimization** If all the border-preserving principal minors of order \( k > R \) have sign \((-1)^R\), then \( f \) has a minimum at \( x^* \) subject to the \( R \) constraints \( g = 0 \).

Equivalently, writing out the conditions \( k > R \) :
Maximization  If the border-preserving principal minors of order $k$ have sign $(-1)^k$, $k = R+1, R+2, \ldots$ then $f$ has a maximum at $x^*$ subject to the $R$ constraints $g = 0$.

Minimization  If the border-preserving principal minors of order $k$ have sign $(-1)^R$, $k = R + 1, R + 2, \ldots$, then $f$ has a minimum at $x^*$ subject to the $R$ constraints $g = 0$.

4.2 Second-order conditions: formulation 2

If there are $R$ equality constraints, we are interested in the minors of orders $k = R+1, k = R+2, k = R+3, \ldots$. In an $R$-constraint problem, a border-preserving principal minor of order $k$ is the determinant of a $(k+R) \times (k+R)$ square matrix. So a principal minor of order $R+1$ is the determinant of a $(R+1+R) = 1+2R$ square matrix. A principal minor of order $R + 2$ is the determinant of an $(R + 2 + R) = 2 + 2R$ square matrix. And a principal minor of order $R + 3$ is the determinant of a $3 + 2R$ matrix. Hence we are interested in the signs of the determinants of matrices of sizes $1 + 2R, 2 + 2R, \ldots$, remembering that to form those matrices we may not delete entire rows and columns containing the border.

Let $(x^*, \lambda^*)$ be a stationary point satisfying the FOCs. Suppose we delete a bunch of rows and corresponding columns of the (bordered) Hessian (without deleting an entire border row/column), so that we are looking at an $m \times m$ square matrix. This is a (border-preserving) minor of some order. What sign must it have in order to satisfy the second-order condition? The following table lists the signs of these determinants for minimization and maximization problems:

<table>
<thead>
<tr>
<th>Sign of $m \times m$ determinant</th>
<th>Number of Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(-1)^m$</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{m-1}$</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{m-R}$</td>
</tr>
<tr>
<td></td>
<td>$m = 1, 2, \ldots, N$</td>
</tr>
<tr>
<td></td>
<td>$m = 3, 4, \ldots, N+1$</td>
</tr>
<tr>
<td></td>
<td>$m = 1 + 2R, \ldots, N + R$</td>
</tr>
<tr>
<td>Minimum</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(-1)^0 = +1$</td>
</tr>
<tr>
<td></td>
<td>$(-1)^1 = -1$</td>
</tr>
<tr>
<td></td>
<td>$(-1)^R$</td>
</tr>
</tbody>
</table>