Introduction to Welfare Economics, II

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1 Introduction

This lecture note continues the previous discussion, this time focussing on problems which break the link between Pareto-Optimality and a Competitive Equilibrium. As before, we work with the 2-good, 2-input, 2-output economy, but note that this is purely for expository convenience, and no results depend on it.

2 Externalities

We say that an externality is present when the actions of one agent affect the well-being of another, and those effects are not reflected in the price system.

2.1 An externality in consumption

Return to the pure exchange economy with fixed quantities of two consumption goods \( \bar{x}_1 \) and \( \bar{x}_2 \). In the original model utility functions were specified as \( u_i(x_{1i}, x_{2i}) \) — that is, \( i \)'s utility depended only on what \( i \) consumed. But now suppose that \( 1 \)'s utility depends also on \( 2 \)'s consumption of good 1: that is, we have \( u_1(x_{11}, x_{12}, x_{21}) \); and for simplicity, we suppose that this is the only interaction. For example, my utility increases the more frequently you paint your house (because it improves general neighborhood property values): here \( u_{13} = \frac{\partial u_1}{\partial x_{21}} > 0 \) and we have a positive externality. Alternatively, suppose my utility decreases the louder you play your stereo: here \( u_{13} = \frac{\partial u_1}{\partial x_{21}} < 0 \) and we have a negative externality.
As before, the Pareto-Optimum in a pure exchange economy maximizes 1’s utility subject to a fixed level $\bar{u}_2$ of 2’s utility, and solves
\[ \mathcal{L} = u_1(x_{11}, x_{12}, \bar{x}_1 - x_{11}) + \lambda(\bar{u}_2 - u_2(\bar{x}_1 - x_{11}, \bar{x}_2 - x_{12})). \]
The FOCs are
\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial x_{11}} &= 0 : \quad u_{11} - u_{13} + \lambda u_{21} = 0 \\
\frac{\partial \mathcal{L}}{\partial x_{12}} &= 0 : \quad u_{12} + \lambda u_{22} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0 : \quad \bar{u}_2 - u_2(\bar{x}_1 - x_{11}, \bar{x}_2 - x_{12})
\end{align*}
\]
where $u_{13} = \partial u_1/\partial x_{21}$. From the first two we get
\[ \frac{u_{11} - u_{13}}{u_{12}} = \frac{u_{21}}{u_{22}} \]
or
\[ \frac{u_{11}}{u_{12}} - \frac{u_{13}}{u_{12}} = \frac{u_{21}}{u_{22}} \]

In the competitive equilibrium, each price-taking agent maximizes utility subject to his/her budget constraint. The dependence of 1’s utility on 2’s consumption is not reflected in the price system, so 1 behaves as if its choice of $x_{21}$ is not under its control, so the decision-variables for 1 are, as before, $x_{11}$ and $x_{12}$ (and not $x_{21}$). In other words, the marginal conditions for the Competitive Equilibrium are exactly as before, that is
\[ \frac{u_{11}}{u_{12}} = \frac{u_{21}}{u_{22}} \]
and the Competitive Equilibrium is no longer a Pareto-Optimum.

### 2.2 An externality in production

Return to the pure production economy, and suppose that firm 1’s productivity depends on its own input bundle and on firm 2’s use of resource 1. That is, 1’s production function is $f_1(z_{11}, z_{12}, z_{21})$. As before, we take no position a priori on the sign of $f_{13} \equiv \partial f_1/\partial z_{21}$. We might have $f_{13} > 0$ in which case $z_{21}$ makes firm 1 more productive (eg, firm 1 is a commercial flower-raiser and 2 is a bee-keeper: the bees from 2 help pollinate the flowers of 1): this is a positive externality. Or we
might have $f_{13} < 0$ in which case, $z_{21}$ has a negative impact on $f_1$ (eg: firm 1 is a laundry which hangs its clean sheets out in the open air to dry; and 2 burns leaves, the smoke from which dirties the laundry hung out next door), which is a negative externality.

Assuming that both final goods are really “good” (ie desirable by consumers) and that we have fixed total quantities of the two resources ($\tilde{z}_1$ and $\tilde{z}_2$), the Pareto-Optimum maximizes the output of firm 1 subject to fixed production of firm 2, ie solves

$$L = f_1(z_{11}, z_{12}, \tilde{z}_1 - z_{11}) + \lambda(\tilde{x}_2 - f_2(\tilde{z}_1 - z_{11}, \tilde{z}_2 - z_{22}))$$

whose FOCs are

$$\frac{\partial L}{\partial z_{11}} = 0 : f_{11} - f_{13} + \lambda f_{21} = 0$$
$$\frac{\partial L}{\partial z_{12}} = 0 : f_{12} + \lambda f_{22} = 0$$
$$\frac{\partial L}{\partial \lambda} = 0 : \tilde{x}_2 - f_2(\tilde{z}_1 - x_{11}, \tilde{z}_2 - x_{12})$$

or (from the first two FOCs)

$$\frac{f_{11} - f_{13}}{f_{12}} = \frac{f_{11}}{f_{12}} - \frac{f_{13}}{f_{12}} = \frac{f_{21}}{f_{22}}$$

or

$$\frac{f_{11}}{f_{12}} = \frac{f_{21}}{f_{22}} + \frac{f_{13}}{f_{12}}$$

(1)

In the competitive equilibrium, each firm maximizes profits as a price taker; however, the dependence of 1’s output on 2’s action (via $z_{21}$) isn’t reflected in the price system, hence is not a decision variable for firm 1. Thus, the marginal conditions for the competitive equilibrium are precisely those we derived earlier, namely

$$\frac{f_{11}}{f_{12}} = \frac{f_{21}}{f_{22}}$$

and the competitive equilibrium no longer appears as a Pareto Optimum.

### 2.3 Internalizing Externalities

How can we cope with externalities and make price-taking optimizing activity by individuals yield a Pareto-Optimum? We continue the production externality story,
and consider two possibilities.

First, suppose that the same profit-maximizing price-taking rm produced both goods (i.e., that the two firms merged: we suppose that they are each small enough so that the merger does not affect their price-taking behavior), and that there are no legal/regulatory barriers to merger. Then the new rm would solve the problem: choose \( z_{11} \) and \( z_{12} \) to maximize joint profits. That is, it maximizes

\[
\Pi = p_1 f_1(z_{11}, z_{12}, z_{21}) + p_2 f_2(z_{21}, z_{22}) - r_1(z_{11} + z_{21}) - r_2(z_{12} + z_{22})
\]

for which the FOCs are

\[
\begin{align*}
\frac{\partial \Pi}{\partial z_{11}} &= 0 : \quad p_1 f_{11} - r_1 = 0 \\
\frac{\partial \Pi}{\partial z_{12}} &= 0 : \quad p_1 f_{12} - r_2 = 0 \\
\frac{\partial \Pi}{\partial z_{21}} &= 0 : \quad p_1 f_{13} + p_2 f_{21} - r_1 = 0 \\
\frac{\partial \Pi}{\partial z_{22}} &= 0 : \quad p_2 f_{22} - r_2 = 0
\end{align*}
\]

so that

\[
\begin{align*}
\frac{f_{11}}{f_{12}} &= \frac{r_1}{r_2} \\
p_1 f_{13} + \frac{f_{21}}{f_{22}} &= \frac{r_1}{r_2}
\end{align*}
\]

and since from conditions (3) and (5), \( (p_1/p_2)(f_{12}/f_{22}) = 1 \), hence \( p_1/p_2 = f_{22}/f_{12} \), the displayed conditions are

\[
\begin{align*}
\frac{f_{11}}{f_{12}} &= \frac{r_1}{r_2} = \frac{f_{22}}{f_{12}} \frac{f_{13}}{f_{22}} + \frac{f_{21}}{f_{22}} \\
\frac{f_{11}}{f_{12}} &= \frac{f_{13}}{f_{12}} + \frac{f_{21}}{f_{22}}
\end{align*}
\]

which are precisely the conditions for Pareto-Optimality. It’s clear what’s going on here: when the same firm produces both goods, it has the ability and interest to take account of the effects of \( z_{21} \) on the output of both of them. What was previously outside the scope of one firm has been brought inside: we say that the externality has been internalized.

Second, consider a taxation solution. The idea here is to make rm 2 aware via the price system of its external impact on rm 1. Specifically, we impose a tax of \( t \)
on each unit of use of $z_{21}$ by firm 2: thus firm 2 solves the problem: choose $z_{21}, z_{22}$ to maximize

$$\Pi_2 = p_2 f_2(z_{21}, z_{22}) - (r_1 + t)z_{21} - r_2 z_{22}$$

whose FOCs are

$$\frac{\partial \Pi}{\partial z_{21}} = p_2 f_2(r_1 + t) = 0$$
$$\frac{\partial \Pi}{\partial z_{22}} = p_2 f_2(r_2) = 0$$

or

$$\frac{f_{21}}{f_{22}} = \frac{r_1}{r_2} + \frac{t}{r_2}$$
$$\frac{f_{21}}{f_{22}} - \frac{t}{r_2} = \frac{r_1}{r_2}$$

In the competitive equilibrium firm 1 maximizes its “ordinary” profits, so, just as before, its decision satisfies $f_{11}/f_{12} = r_1/r_2$. Then under this taxation scheme, with both firms facing the same prices $r_1$ and $r_2$

$$\frac{f_{21}}{f_{22}} - \frac{t}{r_2} = \frac{f_{11}}{f_{12}}.$$ 

In order to correctly internalize the externality (make the equilibrium realizable as an optimum) the tax must equate these conditions to the marginal conditions of the optimum (1). Thus

$$\frac{f_{21}}{f_{22}} - \frac{t}{r_2} = \frac{f_{11}}{f_{12}} = \frac{f_{21}}{f_{22}} + \frac{f_{13}}{f_{12}}$$

so

$$- \frac{t}{r_2} = \frac{f_{13}}{f_{12}}$$

or

$$t = -r_2 \frac{f_{13}}{f_{12}}$$

Note how this works. Assume $f_{12} > 0$. Then if $f_{13} > 0$, $t$ is negative, ie will be a subsidy. That is, a subsidy to firm 2 is called for when the external impact of firm 2’s decisions on firm 1 is positive — ie enhances 1’s productivity. The idea in this case is to encourage firm 2 to use more of input 1 by reducing the price it pays for this input. When $f_{13} < 0$ we get $t$ is positive, with precisely the opposite motivation: here we want to induce firm 2 to do “less of a bad thing”.

5
Note that these taxes will not arise on their own. Some sort of government intervention by a market authority (regulator, planner) will be needed to set the appropriate level. And note too the information that such an authority needs: it must know the marginal products of firm 1. As a practical matter, this may require a good deal of empirical study.

3 Public Goods

A *public good* is one which can be simultaneously consumed by everyone, with no loss in quantity or quality. Put another way, once the public good is produced, it is impossible to exclude anyone from consuming it, and such consumption does not detract from the quantity available to others. Examples: national defense, TV or radio. The fact that you and I simultaneously watch a program on WOSU does not detract from the signal quality available to either of us.

To see the impact of the existence of public goods, we return to the production and exchange economy, and we suppose that $x_2$ is a public good. There are fixed quantities of the two resources available, denoted $\tilde{z}_1$ and $\tilde{z}_2$; and the Pareto-Optimum solves the problem: choose $x_{11}, x_{21}, x_1, x_2, z_{11}$ and $z_{12}$ to maximize

$$
\mathcal{L} = u_1(x_{11}, x_2) + \lambda_1(\tilde{u}_2 - u_2(x_{21}, x_2))
+ \lambda_2(x_1 - f_1(z_{11}, z_{12})) + \lambda_3(x_2 - f_2(\tilde{z}_1 - z_{11}, \tilde{z}_2 - z_{12}))
+ \lambda_4(x_1 - x_{11} - x_{21})
$$

Note in this formulation *both* utility functions depend on the entire quantity of $x_2$ produced, and therefore there is no need for an adding-up constraint on $x_2$, since it is not divided between the two consumers. To put it another way, each consumer consumes *all* of $x_2$ (watches the same TV program) simultaneously.
The FOCs are

\[
\begin{align*}
\frac{\partial L}{\partial x_{11}} &= 0 : \quad u_{11} - \lambda_4 = 0 \\
\frac{\partial L}{\partial x_{21}} &= 0 : \quad -\lambda_1 u_{21} - \lambda_4 = 0 \\
\frac{\partial L}{\partial x_2} &= 0 : \quad u_{12} - \lambda_1 u_{22} + \lambda_3 = 0 \\
\frac{\partial L}{\partial x_1} &= 0 : \quad \lambda_2 + \lambda_4 = 0 \\
\frac{\partial L}{\partial z_{11}} &= 0 : \quad -\lambda_2 f_{11} + \lambda_3 f_{21} = 0 \\
\frac{\partial L}{\partial z_{12}} &= 0 : \quad -\lambda_2 f_{11} + \lambda_3 f_{22} = 0
\end{align*}
\]

(plus FOCs for the Lagrange multipliers). Conditions (6) and (8) may be written

\[
\frac{u_{12}}{u_{11}} = \frac{\lambda_1}{\lambda_4} u_{22} - \frac{\lambda_3}{\lambda_4}
\]

and from condition (7)

\[
\frac{\lambda_1}{\lambda_4} = -\frac{1}{u_{21}}
\]

so we have

\[
\frac{u_{12}}{u_{11}} + \frac{u_{22}}{u_{21}} = -\frac{\lambda_3}{\lambda_4}
\]

Then, from condition (9), \( \lambda_2 = -\lambda_4 \) and from (10),

\[
\frac{f_{11}}{f_{21}} = \frac{\lambda_3}{\lambda_2} = -\frac{\lambda_3}{\lambda_4}
\]

so, finally, the PO requires

\[
\frac{u_{12}}{u_{11}} + \frac{u_{22}}{u_{21}} = \frac{f_{11}}{f_{21}}
\]

The MRT \((f_{11}/f_{21})\) equals the sum of the marginal rates of substitution by each consumer. This is to be contrasted with the conditions which will obtain under separate utility maximization (the competitive equilibrium), namely

\[
\begin{align*}
\frac{u_{12}}{u_{11}} &= \frac{f_{11}}{f_{21}} \\
\frac{u_{22}}{u_{21}} &= \frac{f_{11}}{f_{21}}
\end{align*}
\]

— that is, the MRT will equal each of the MRS’s separately. Once again, the PO is not realized as a CE.
4 Theory of the Second Best

We continue to work with the (original form of the) production and exchange economy; but no we suppose that — for whatever reason — one of the marginal conditions for the PO cannot be satisfied. For example, suppose that instead of 

\[ u_{12}/u_{11} = u_{22}/u_{21} \]

we have

\[ u_{12}/u_{11} = k(u_{22}/u_{21}) \]

for some \( k \neq 1 \). In general, this will be an additional constraint of the form

\[ g(x_{11}, x_{12}, \bar{x}_1 - x_{11}, \bar{x}_2 - x_{12}) = 0 \]

The PO solves

\[
\mathcal{L} = u_1(x_{11}, x_{12}) + \lambda_1(\bar{u}_2 - u_2(x_{11}, x_{21} - x_{12})) \\
+ \lambda_2(x_1 - f_1(z_{11}, z_{12})) + \lambda_3(x_2 - f_2(z_1 - z_{11}, z_2 - z_{12})) \\
+ \lambda_4 g(x_{11}, x_{12}, x_1 - x_{11}, x_2 - x_{12})
\]

whose FOCs in the \( x \)’s are

\[
\begin{align*}
    u_{11} + \lambda_1 u_{21} + \lambda_4 (g_1 - g_3) &= 0 \\
    u_{12} + \lambda_1 u_{22} + \lambda_4 (g_2 - g_4) &= 0
\end{align*}
\]

The critical thing to note here is that all of these involve (derivatives of) \( g \), and none of these derivatives will appear in the competitive equilibrium. So the failure of the conditions for the PO to hold in just one sector means that the competitive equilibrium is no longer a PO in a global sense.

5 Imperfect Competition

So far we have assumed that individuals and firms are price-takers in markets: that is, they can buy or sell any quantities without affecting prices. Imperfect competition will exist when this is not so: that is, when prices vary in response to supply or demand conditions.

Return to the pure exchange economy, but now suppose that the price of good 1 depends on the total demand according to \( p_1 = g(x_{11} + x_{21}) \), with \( g' < 0 \). The conditions for a Pareto Optimum do not depend on the form of economic organization, so they are as before.
What about the equilibrium? (Note that it is not a competitive equilibrium). Individual $i$ maximizes
\[ L = u_i(x_{i1}, x_{i2}) + \lambda_i(M_i - g(x_{i1} + x_{i2})x_{i1} - p_2x_{i2}) \]
and we obtain the FOCs
\[
\begin{align*}
    u_{i1} - \lambda_i(g'x_{i1} - g) &= u_{i1} - \lambda_i(g'x_{i1} + p_1) = 0 \\
    u_{i2} - \lambda_ip_2 &= 0 \\
    M_i - g(x_{i1} + x_{i2})x_{i1} - p_2x_{i2} &= 0
\end{align*}
\]
(since in the first equation $g(x_{i1} + x_{i2}) = g = p_1$). From the first and second of these
\[
\frac{u_{i1}}{u_{i2}} = \frac{g'x_{i1} + p_1}{p_2} = \frac{g'x_{i1}}{p_2} + \frac{p_1}{p_2}
\]
So for our 2-consumer economy the equilibrium conditions are
\[
\begin{align*}
    \frac{u_{i1}}{u_{i2}} - \frac{g'x_{i1}}{p_2} &= \frac{p_1}{p_2} = \frac{u_{i21}}{u_{i22}} - \frac{g'x_{i21}}{p_2} \\
    \frac{u_{i1}}{u_{i2}} - \frac{g'x_{i1}}{p_2} &= \frac{u_{i21}}{u_{i22}} - \frac{g'x_{i21}}{p_2}
\end{align*}
\]
and these violate the conditions for Pareto-Optimality, which, as always, are
\[
\frac{u_{i1}}{u_{i2}} = \frac{p_1}{p_2}
\]
for all individuals $i$, so, here
\[
\frac{u_{i1}}{u_{i2}} = \frac{u_{i21}}{u_{i22}}
\]
Similar results obtain for the pure production and production+exchange economies. So it’s not the fact of equilibrium that’s important for considerations of welfare: it’s that it’s a competitive equilibrium which matters.

6 Regulation and IRTS

Under competition, a price-taking profit-maximizer will price at marginal cost, and, as we’ve seen repeatedly, this is the welfare (Pareto) optimum. But suppose
that production takes place under IRTS. Then (a) any firm has an incentive to expand production, in order to reap the benefits of lower average cost; (b) therefore competition, in the sense of large numbers of firms, is infeasible — the industry is a natural monopoly; and (3) marginal-cost pricing, even if the firm wanted — or was induced by market forces — to implement it, is infeasible, since under IRTS, $AC < MC$. Competition will not work. What can be done?

One solution is to regulate the firm which produces under IRTS. Historically, there are two ways to do so. The first is called rate-of-return regulation: the firm is permitted to select its inputs on its won as long as it earns no more than fixed return on its capital (the rate set by the regulator). The second directly sets the prices the firm may charge.

### 6.1 Rate-of-Return Regulation and the A-J Effect

In this section we consider the effect of rate-of-return regulation. We make one additional assumption purely for ease of exposition, namely that the firm is a price-taker in the output market. Clearly this is wrong: but you get the same result with a little more work if you relax it.

Suppose then that the firm produces a single output according to the 2-input production function $f(K, L)$ where $K$ is the capital stock and $L$ is the quantity of labor and the production function satisfies $f_K, f_L > 0, f_{KK}, f_{LL} < 0$ and $f_{KK} f_{LL} - f_{KL}^2 > 0$ (isoquants have the usual shapes). The firm is a price-taker in the output market (output price is $p$) and in the input markets, where the price of capital is $r$ and the price of labor is $w$. The firm’s return on capital is total revenue minus labor cost; the rate of return on capital is the return on capital per unit of capital stock; and we assume that this return is regulated to be exactly equal to $s > r$. That is, the firm is permitted a return on capital exceeding the market return. The firm maximizes profits

$$pf(K, L) - rK - wL$$

subject to the constraint

$$\frac{pf - wL}{K} = s$$

or

$$pf - wL = sK$$
The Lagrangian is
\[ \mathcal{L} = pf(K, L) - rK - wL + \lambda (sK - pf(K, L) + wL) \]
and the FOCs are
\[ \frac{\partial \mathcal{L}}{\partial K} = 0 : \quad pf_K - r + \lambda (s - pf_K) = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial L} = 0 : \quad pf_L - w + \lambda (pf_L + w) = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 : \quad sK - pf(K, L) + wL = 0 \]
or
\[ pf_K (1 - \lambda) = r - \lambda s \]
\[ pf_L (1 - \lambda) = w(1 - \lambda) \]
\[ sK = pf(K, L) + wL \]

Now, it is easy to show that \( 0 < \lambda^* < 1 \): first, by the envelope theorem, \( \partial \Pi^* \partial s = \lambda^* K^* \) and this should be \( > 0 \), since increasing the allowable rate of return should increase profits. Hence, since \( K^* > 0 \), we have \( \lambda^* > 0 \). Similarly, \( \partial \Pi^*/\partial p > 0 \), since increasing output price should increase profits. Then \( \partial \Pi^*/\partial p = f^* - \lambda f^* = f^*(1 - \lambda^*) \). Since \( f^* > 0 \), then \( 1 - \lambda^* > 0 \). Therefore \( 0 < \lambda^* < 1 \).

Now look again at the first FOC
\[ pf_K (1 - \lambda) = r - \lambda s \]
\[ = r - \lambda r - \lambda s + \lambda r \]
\[ = r(1 - \lambda) + \lambda (r - s) \]
and combining this with the second FOC we get
\[ \frac{pf_K}{pf_L} = \frac{r(1 - \lambda) + \lambda (r - s)}{w(1 - \lambda)} \]
\[ \frac{f_K}{f_L} = \frac{r}{w} + \frac{\lambda}{1 - \lambda} \frac{r - s}{w} \]
Now \( \lambda/1 - \lambda > 0 \), \( r - s/w < 0 \) (since we are assuming that \( s > r \), and of course we know that \( w > 0 \)), so this is
\[ \frac{f_K}{f_L} = \frac{r}{w} + y \]
(with \( y < 0 \)) or, multiplying through by \(-1\),

\[-\frac{f_K}{f_L} = -\frac{r}{w} + y\]

What does this say? That under regulation, the MRT (slope of the isoquant, ie \(-f_K/f_L\)) will be more than \(-r/w\), which is what it would be when the firm is unregulated. So the firm’s utilization of capital under rate-of-return regulation is higher than it would be otherwise: regulation induces the firm to over-use capital. This is the well-known Averch-Johnson (A-J) Effect.

6.2 Quasi-Optimal Pricing

We now discuss the second approach to regulation of a natural monopoly, direct regulation of prices. We assume that the regulator is interested in setting prices to maximize community welfare, while at the same time assuring that revenues are sufficient to just cover costs for the producer. In effect, this is the standard US solution to the problem: regulate the market. In Europe, at least until recently, the natural solution would probably have been nationalization (which this note does not discuss).

We work with a population of identical price-taking individuals, and in particular, with a representative member of the population. We focus on the set of \( N \) goods \( x_1, \ldots, x_N \) produced by the firm: the (Marshallian) demand for this good by the representative individual is \( x_i^*(p, M) \) where \( p \) is a vector of consumer prices and \( M \) is the individual’s income. The welfare of this individual is measured by the indirect utility function \( V(p, M) \). The firm produces the \( N \) products according to the per-capita cost function \( C^*(x_1^*(p, M), x_2^*(p, M), \ldots, x_N^*(p, M)) \equiv C^*(x^*(p, M)) \). Then the regulator’s problem is to select the price vector \( p \) to maximize \( V(p, M) \) subject to the constraint that revenues exactly cover costs. The Lagrangian is

\[ L = V(p, M) + \lambda \left( \sum_i p_i x_i^* - C^*(x^*(p, M)) \right) \]

leading to the FOC

\[ \frac{\partial L}{\partial p_j} = 0 \quad : \quad \frac{\partial V}{\partial p_j} + \lambda \left( x_j^* + \sum_i \left( \frac{\partial x_i^*}{\partial p_j} - \frac{\partial C^*}{\partial x_i} \frac{\partial x_i^*}{\partial p_j} \right) \right) = 0 \]

(plus the constraint).
Now, by Roy’s Identity,
\[
\frac{\partial V}{\partial p_j} = -x^*_j
\]
so that writing \( \theta \equiv \frac{\partial V}{\partial M} \) (the marginal utility of income) we have \( \frac{\partial V}{\partial p_j} = -\theta x^*_j \) and
\[
0 = -\theta x^*_j + \lambda \left( x^*_j + \sum_i \left( p_i \frac{\partial x^*_i}{\partial p_j} - \frac{\partial C^*_i}{\partial x^*_i} \right) \right)
\]
\[
\frac{\theta}{\lambda} x^*_j = x^*_j + \sum_i \left( p_i \frac{\partial x^*_i}{\partial p_j} - \frac{\partial C^*_i}{\partial x^*_i} \right)
\]
\[
x^*_j \mu = \sum_i \left( p_i - C^*_i \right) \frac{\partial x^*_i}{\partial p_j}
\]
where \( \mu = (\theta/\lambda) - 1 = (\theta - \lambda)/\lambda \) and \( C^*_j \equiv \frac{\partial C^*}{\partial x^*_j} \), the marginal cost of producing good \( j \).

We now make a special-case assumption, in order to get an idea of what is going on. Specifically, we shall assume that the demands are independent, that is, that \( x^*_i \) depends on \( p_i \) and on none of the other prices. Then \( \frac{\partial x^*_i}{\partial p_j} = 0 \) for \( j \neq i \), and
\[
x^*_j \mu = \left( p_j - C^*_j \right) \frac{\partial x^*_i}{\partial p_j}
\]
Now divide through by \( x^*_j \) and multiply and divide on the right by \( p_j \). The result is
\[
\mu = \left( \frac{p_j - C^*_j}{p_j} \right) \frac{\partial x^*_j}{\partial p_j} \frac{p_j}{x^*_j}
\]
We recognize the term to the right of the brackets as the own-price demand elasticity for good \( j \), \( \eta_{jj} \), and the term in the brackets as the percent deviation of price and marginal cost. The final result may be written in two ways: first, writing \( \%\Delta_j = \left( p_j - C^*_j \right) / p_j \) the previous display is
\[
\mu = \%\Delta_j \eta_{jj}
\]
so
\[
\frac{\mu}{\eta_{jj}} = \%\Delta_j.
\]
We see that in order to maximize consumer welfare under the condition that revenues to the firm cover costs, the percent deviation of price and marginal cost
should be \textit{inversely proportional to the own-price elasticity of demand} — this is the famous Inverse Elasticity Rule. In other words, the percent deviation of price and marginal cost should be \textit{higher} for goods with inelastic demands than for goods with elastic demands. (Note that this simple form depends on the independence assumption: without it, the rule is more complicated, but basically says the same thing).

Second, for any two goods $j$ and $k$ we have

\[
\% \Delta_j \cdot \eta_{jj} = \% \Delta_k \cdot \eta_{kk}
\]

which says that the percentage deviations of price and marginal cost, weighted by the own-price elasticities of demand, must be \textit{equal} for any two goods: this is the equally famous Ramsey-Price Rule. (Again, in the dependent-demand case the rule is more complicated).

In practice, regulatory agencies (for example, the now deceased US Civil Aeronautics Board which regulated airlines, or the Interstate Commerce Commission which regulated interstate surface transportation, or the Federal Power Commission, or state regulatory authorities like Ohio’s PUCO, spend a lot of their time and resources trying to determine empirically the components of equations like these: marginal costs and demand elasticities, precisely in order to be able to set what are referred to as \textit{quasi-optimal} prices (ie prices which maximize social welfare subject to the constraint of market viability).