Present Value Concepts

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Many projects lead to impacts that occur at different times.

We will refer to those impacts as constituting an (inter)temporal stream.

These will be net impacts, with positive elements being benefits, and negative ones, costs.

Our problem is, when we see different projects with different streams of impacts, how can we decide which is better?
Consider the following two projects, which have net impacts in only three periods:

<table>
<thead>
<tr>
<th></th>
<th>This year</th>
<th>Next Year</th>
<th>Year After</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project A</td>
<td>50</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Project B</td>
<td>10</td>
<td>20</td>
<td>50</td>
</tr>
</tbody>
</table>

If you could undertake only one of these, do you have a preference?
Most people, I think, would prefer project A, since it delivers its benefits sooner than project B.

But irrespective of this, if you have some preference — you do not think that the question should be settled by flipping a fair coin — then we learn something important here.

Note that the numerical sum of the impacts of the two projects is the same (80).

So if you do not regard the two as equally desirable, then we have shown that it is incorrect to compare the two by simply adding up their impacts over time (since that would rate them as equal).

Time matters: $100 at one time and $100 at another time are not the same thing, and cannot consistently be added together.
To develop this, suppose you are in a situation where your money can earn 5% per year, for example at a bank, and that this is the best rate you can get.

And suppose that I offer you $100 now.

You know that in one year, if you deposit it in the bank, it will grow to $100 \times 1.05 = 105$.

Therefore, you should be indifferent between getting $100 now and $105 in one year, since either way, you can end up with $105 next year.
Let's formalize this.

Since we need to refer to money amounts (impacts) by the year in which they are received let’s write $P_t$ for $P$ received in year $t$.

And let’s write $t = 0$ for “today”, $t = 1$ for “next year” etc.

Then our example says that you should be indifferent between (regard as equally good) $P_0 = 100$ received today, and $P_1 = 100 \times 1.05 = 105 = 1.05P_0$ received next year.

So if we interpret “=” to mean “indifferent between” then we have

$$P_0(1.05) = P_1$$
Now let’s reverse the story.

Suppose I offer you the possibility of receiving \( P_1 \) one year from now.

What amount \( P_0 \) received now would be equivalent to that?

Well, as long as you can get interest at 5% and since:

\[
P_1 = (1.05)P_0
\]

the answer is obviously:

\[
P_0 = \frac{P_1}{1.05}
\]

Thus 105 in year 1 is equivalent to \( 105/1.05 = 100.0 \) today.
We say that the *Present Value* of $105 received in year 1 is $100.

In general, as long as your best interest rate is 5%, then you should be indifferent between:

- receiving $P_1$ next year; and
- receiving its present value $P_0 = P_1 / 1.05$ now
If you take $100 and leave it on deposit for two years, then, assuming that the interest rate doesn’t change, at the end of this period you’ll have:

\[ P_2 = [100 \times 1.05] \times 1.05 = P_0 (1.05)^2 \]

because in one year your 100 will grow to 100 \times 1.05, and in one year more the \([100 \times 1.05]\) will continue to earn interest at 5%, and hence will grow to \([100 \times 1.05] \times 1.05 = P_0 (1.05)^2\).
Let’s work backwards once again
If I offer you $P_2$ in year 2, what quantity $P_0$ received now would be equivalent to that?
Obviously, since:

$$P_2 = P_0 (1.05)^2$$

we have:

$$P_0 = \frac{P_2}{1.05^2}$$

We say that the present value of $P_2$ (received in year 2) is $P_2/(1.05)^2$.
Example: if you can earn 5% per year then the present value of $100 to be received in year 2 is $100/1.05^2 = 100/1.1025 = 90.703$. 
Let’s generalize this.

First, suppose that the best interest rate you can earn is \( r \). We will assume that this is expressed as a decimal, so that when we say that you can get an interest rate of 5\%, this will imply that \( r = 0.05 \).

Then our previous arguments imply that the present value of \( P_1 \) (received in year 1) is:

\[
\frac{P_1}{(1 + r)}
\]

and the present value of \( P_2 \) received in year 2 is:

\[
\frac{P_2}{(1 + r)^2}
\]
We can see where this is going.

$P_0$ deposited now and left on deposit for three years at an annual interest rate of $r$ will grow (assuming no inflation) to $P_0(1 + r)^3$.

So the present value of $P_3$ is $P_3/(1 + r)^3$.

And the present value of a quantity $P_t$ received in year $t$ is

$$PV(P_t) = \frac{P_t}{(1 + r)^t}$$

This is our fundamental result
Main Result

- Suppose a project provides a net impact of $P_t$ in year $t$.
- Then (assuming no change in the interest rate $r$ over time), you should be indifferent between:
  - receiving $P_t$ in period $t$; and
  - receiving its present value, $P_t(1 + r)^t$ now
- That’s pretty much all there is to the idea of present value.
- The present value calculation converts an amount $P_t$ in period $t$ to an equivalent amount received now.
Let's return to our initial examples.

There we had two projects, generating temporal streams of net impacts, and our question was, which of them is better.

Armed with our new tools, we now have a way of answering that question.

We replace each period’s temporal impact by its present value, i.e. the equivalent amount received now (at time $t = 0$).

Then we add up the present values: this is legitimate since all present values are year-0 quantities.
For Project A we have:

<table>
<thead>
<tr>
<th>Project A</th>
<th>This year</th>
<th>Next Year</th>
<th>Year After</th>
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<tr>
<td>Impact</td>
<td>50</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

We write down the present value of each impact below the original. Let’s suppose that the bank pays us 1% per annum so that \( 1 + r = 1.01 \). Then:

<table>
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<tr>
<th>Project A</th>
<th>This year</th>
<th>Next Year</th>
<th>Year After</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impact</td>
<td>( \frac{50}{(1.01)^0} )</td>
<td>( \frac{20}{(1.01)^1} )</td>
<td>( \frac{10}{(1.01)^2} )</td>
</tr>
<tr>
<td>Calculation</td>
<td>50</td>
<td>19.606</td>
<td>9.8030</td>
</tr>
</tbody>
</table>

So for the entire project A we have:

\[
PV(\text{project A}) = 50 + 19.606 + 9.8030 = 79.409
\]
For Project B we have:

<table>
<thead>
<tr>
<th>Project B</th>
<th>This year</th>
<th>Next Year</th>
<th>Year After</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impact</td>
<td>10</td>
<td>20</td>
<td>50</td>
</tr>
<tr>
<td>Calculation</td>
<td>(10/(1.01)^0)</td>
<td>(20/(1.01)^1)</td>
<td>(50/(1.01)^2)</td>
</tr>
<tr>
<td>(PV(\text{impact}))</td>
<td>10</td>
<td>19.606</td>
<td>49.015</td>
</tr>
</tbody>
</table>

So for the entire Project B we have:

\[
PV(\text{project B}) = 10 + 19.606 + 49.015 = 78.621
\]

and comparing the two projects by their present values, we conclude that Project A is the better one.
A project that delivers the same net impact over a period of years is called an annuity (over that period).

Example: a project that provides net impacts of 10 over years $t = 2$ to $t = 50$ is an annuity.

In principle, annuities are no different from “regular” projects: we convert each year’s impact to its year-0 equivalent (present value) and then add up the present values.

But it turns out that we can take advantage of the structure of an annuity to simplify the calculation of its present value.
Suppose our project delivers a net impact of $P$ during periods $t_1$ to $T$ (inclusive).

The it turns out that, if we define $s = 1/(1 + r)$, the present value of the annuity is:

$$PV(\text{annuity}) = P \frac{s^{t_1} - s^{1+T}}{1 - s}$$

(note the $1 + T$ in the exponent of the right-hand term of the numerator: forgetting the 1 is a common mistake).
Example: a project delivers 100 each year from $t_1 = 4$ to $t_2 = 30$. If $r = 1.01$ then $s = 1/1.01 = 0.99010$, and the present value is

$$PV = 100 \times \frac{0.990\times 10^4 - 0.990 \times 10^{31}}{1 - 0.990 \times 10} = 100 \times \frac{0.960\times 98 - 0.73460}{0.0099} = 100 \times 22.867 = 2286.7$$

Note that the annuity consists of 27 terms, each valued at 100, so the numerical sum is 2700: this is quite a bit different from the present value, 2287.
Suppose that the annuity’s impact begins in period $t_1$ and goes on forever.

An example might be environmental remediation: once we’ve done it (and as long as we don’t do anything untoward), we receive the benefits forever.

What is the present value of this annuity?

To compute this, we need to see what happens to our annuity formula as $T$ (the ending year of the impact stream) gets very (infinitely) large.
To study this, note that $s = 1 / (1 + r)$ is less than one. So we are raising a quantity that is less than one to successively higher powers.

As you can easily convince yourself with a pocket calculator, as $T$ gets very large, the term $s^{1+T}$ gets smaller and smaller, and eventually becomes so small that it can be ignored.

This means that we have, for an infinite annuity beginning in $t_1$:

$$PV(\text{annuity}) = \frac{s^{t_1}}{1 - s}$$

A special case is when the annuity begins now (ie at $t = 0$) and lasts forever. Then we have:

$$PV(\text{annuity}) = \frac{1}{1 - s}$$
Suppose we have computed the present value of a project.

It is sometimes useful, for presentation purposes, to compute an equivalent annuity: this will be a quantity $P$ received each year during the lifetime of the project, such that the present value of the annuity is the same as the present value of the project itself.

Computing the equivalent annuity is known as annualization.

Note that this is just a matter of presentation: annualizing a project’s impacts adds nothing to project evaluation beyond what is contained in the present value: it’s just a matter of how you present the result.
Suppose our project has a present value of $P$ and that the interest rate is $r$.

And suppose the project lasts $T$ years.

Then the annualized value of the project’s impact, the annuity from today ($t = 0$) to $t = T$, is given by

\[
\text{Annualized value} = \frac{(1 - s)P}{1 - s^{T+1}}
\]

where $s = 1/(1 + r)$. 
So far we have modelled time as if it occurred at a series of discrete points, which we could think of as years.

For many project evaluation purposes, this is all we need.

But in some contexts, we may want to go further.

If we ask, for example, what is the best time to do a project, we would like to use the optimization tools provided by calculus; and this requires that time be considered as a continuous quantity.

So we are led to ask: if time is considered as continuous, what is the present value of $P_t$?
To develop this, suppose as before that $r$ is the *annual* rate of interest available to us.

But now suppose that we are paid interest every half-year, rather than every year.

Then the interest rate actually used will be $r/2$, and at the end of the first half-year $P_0$ will grow to:

$$P_0\left(1 + \frac{r}{2}\right)$$

And at the end of the year it will have received a further interest payment, and will therefore have grown to:

$$P_1 = P_0\left(1 + \frac{r}{2}\right)^2.$$
Now suppose that we are paid interest on a quarterly basis. Then each payment will be based on $1 + r/4$, and at the end of one year (4 payments) we will have

$$P_1 = P_0 (1 + \frac{r}{4})(1 + \frac{r}{4})(1 + \frac{r}{4})(1 + \frac{r}{4})$$

$$= (1 + \frac{r}{4})^4$$

What if interest was paid monthly? Then the rate will be $1 + r/12$ and

$$P_1 = P_0 (1 + \frac{r}{12})^{12}$$

And if interest was paid daily we’d see:

$$P_1 = P_0 (1 + \frac{r}{365})^{365}$$
Let's see how this plays out. Suppose that the annual rate is 0.05, and let $P_0 = \$1$. Then

\[
\begin{array}{llll}
\text{Payment rate} & k & 1 + \frac{r}{k} & (1 + \frac{r}{k})^k \\
\hline
\text{Annual} & 1 & 1.05 & 1.05 \\
\text{Semi-annual} & 2 & 1.025 & 1.0506 \\
\text{Quarterly} & 4 & 1.0125 & 1.0509 \\
\text{Monthly} & 12 & 1.0042 & 1.051162 \\
\text{Half-monthly} & 24 & 1.0021 & 1.051216 \\
\text{Daily} & 365 & 1.000137 & 1.051267 \\
\end{array}
\]
Continuous Time (IV)

- We could obviously continue like this, subdividing time even more finely.
- But we notice something interesting: as we do so, differences in the final answer (last column) get smaller and smaller.
- For example, the difference between a situation in which interest is computed daily and in which it is computed every half-month is
  \[1.051267 - 1.051216 = 0.000051\]
- This suggests that we may be able to skip all the intermediate steps and get an answer for the case where interest is being computed (compounded) continuously.
This turns out to be correct. If we allow the number of compoundings per year, $n$, to get infinitely large, then it can be shown that:

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r$$

So in our example, with $r = .05$, the limit case is that $1$ will grow in one year with continuous compounding to:

$$1 \times e^{0.05} = 1.051271$$

not very different from what we got with daily compounding.
Continuous Present Value (I)

- We can now imitate our previous argument. With continuous compounding, $P_0$ will grow in one year to:

$$P_1 = P_0 e^r$$

- Then if we know the year-1 quantity $P_1$, its period-0 equivalent, ie its present value, will be:

$$P_0 = PV(P_1) = \frac{P_1}{e^r} = P_1 e^{-r}$$

- Similarly, you should be able to convince yourself that with continuous compounding, $P_0$ will grow in 2 years to:

$$P_2 = P_0 e^{2r}$$

and thus:

$$PV(P_2) = P_2 e^{-2r}$$
Finally, the general case. If interest is computed continuously and we leave $P_0$ on deposit for $t$ years, it will grow to:

$$P_t = P_0 e^{rt}$$

so that with continuous compounding, the present value of $P_t$ received in year $t$ is:

$$PV(P_t) = P_t e^{-rt}$$
It turns out all the other special cases we developed for the discrete case have formulas for the continuous case too.

The next two slides summarize all formulas, first for the discrete case, and then for the continuous case.
Formulas: Discrete Time

- Present value of $P_t$ received in year $t$:
  \[
  \frac{P_t}{(1 + r)^t}
  \]

- Present value of an annuity of $P$ received in years $t_0$ to $T$:
  \[
  P \left( \frac{s^{t_0} - s^{T+1}}{1 - s} \right) \quad s = 1/(1 + r)
  \]

- Present value of an annuity beginning in year $t_0$ and lasting forever:
  \[
  P \frac{s^{t_0}}{1 - s}
  \]

- Annualized $P_0$ over $T$ years:
  \[
  \frac{(1 - s)P_0}{1 - s^{T+1}}
  \]

- Annualized $P_0$ when impact lasts forever:
  \[
  (1 - s)P_0
  \]
Formulas: Continuous Time

- Present value of $P_t$ received in year $t$:
  \[ P_t e^{-rt} \]

- Present value of an annuity of $P$ received in years $t_0$ to $T$:
  \[ \frac{P}{r} \left( e^{-rt_0} - e^{-rT} \right) \]

- Present value of an annuity beginning in year $t_0$ and lasting forever:
  \[ \frac{P}{r} e^{-rt_0} \]

- Annualized $P_0$ over $T$ years:
  \[ \frac{rP_0}{1 - e^{-rT}} \]

- Annualized $P_0$ when impact lasts goes on forever:
  \[ rP_0 \]
In our work on road maintenance, we are concerned with a recurring impact $P$ which happens every $T$ years, and that lasts forever. For this case, and assuming continuous time, we have

- Present value of $P$ occurring every $T$ years, forever:
  \[
  \frac{P}{e^{rT} - 1}
  \]

- Annualized value of the above quantity (also lasting forever):
  \[
  \frac{rP}{e^{rT} - 1}
  \]
So far we have taken a completely individualistic approach to discounting. But there are a number of reasons why one might wish to take a broader (non-individualistic) view:

- **Technical change**: in a world in which productivity is improving over time, consumption opportunities will go up, irrespective of whether we invest today.

- **Population growth**: if the population is increasing and we do not alter our investment policy today, then (other things being equal) future generations will be worse off than we were.
Other Approaches

- It turns out that taking a non-individualistic approach involves an alternative logic for selecting the appropriate discount rate \((1 + r)\).
- Some possibilities are:
  - The planner wants to focus on intertemporal sustainability: ie achieve a sustainable standard of living for this and all future generations. This leads to the idea of the Social Discount Rate.
  - Another approach is to observe that there is competition for scarce investment funds: they can be used be either the public sector or the private sector. This leads to the idea of the Social Opportunity Cost of Capital.
- See advanced treatments of project evaluation for more on these.
Here is a table of what now (2-12–2013) appear to be the options for the discount rate \((1 + r)\) in the formula for the present value of \(P_t\):

\[
PV(P_t) = \frac{P_t}{(1 + r)^t}
\]

<table>
<thead>
<tr>
<th>Rate Description</th>
<th>10yr</th>
<th>20yr</th>
<th>30yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individualistic</td>
<td>0.9932</td>
<td>1.0004</td>
<td>1.0040</td>
</tr>
<tr>
<td>Social Rate</td>
<td>0.9962</td>
<td>1.0034</td>
<td>1.0070</td>
</tr>
<tr>
<td>Opp. Cost of Capital</td>
<td>1.0105</td>
<td>1.0233</td>
<td>—</td>
</tr>
</tbody>
</table>