Calculating the Gravity Model

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One thing that often gives people problems is the calculation of the denominator of the gravity model. In this note I’ll illustrate what is going on. Consider a simplified version of the gravity model in which we have

\[ T_{ij} = \frac{O_i A_j F_{ij}}{\sum_m A_m F_{im}} \]

—the simplification is that we ignore the zonal adjustment factors (the \( K_{ij} \)), since including these adds nothing essential to the purely computational point.

We are computing the number of trips between a fixed origin, denoted by \( i \), and a fixed destination denoted by \( j \). The denominator tells you to add a series of terms together, terms which are indexed by \( m \). The first thing to understand is that the notation \( \sum_m \) means that \( m \) is to be taken to range over all values from 1 to the number of zones \( Z \). So the denominator can be written equivalently as

\[ \sum_{m=1}^Z A_m F_{im} \]

The value of \( i \) is fixed. This means that we will be adding up a series of terms that looks like this:

\[
A_1 F_{i1} + A_2 F_{i2} + A_3 F_{i3} + \ldots + A_Z F_{iZ}
\]

\((m = 1) \quad (m = 2) \quad (m = 3) \quad \ldots \quad (m = Z)\)

If you look closely at this, you’ll see that all the \( F \) terms are from row \( i \) of the \( F_{ij} \) matrix. So what we do for the denominator of the \( T_{ij} \) term is pick out row \( i \) of \( F_{ij} \) and then multiply it term-by-term with the elements of the attractions vector \( A \); and then add up the result.
Here’s a numerical example. Suppose we have $Z = 3$ zones, with

$$F = \begin{bmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \\ 3.1 & 3.1 & 3.2 \end{bmatrix}, \quad O = [10.1, 10.2, 10.3], \quad A = [20.1, 20.2, 20.3]$$

and suppose that we want to compute the number of trips $T_{23}$ between zones $i = 2$ and $j = 3$ using the gravity model. This is given by

$$T_{23} = \frac{O_2A_3F_{23}}{\sum_m A_m F_{2m}}$$

where we note that we have plugged in the value $i = 2$ into the $F_{im}$ term in the denominator.

The numerator is easy, and we have

$$10.2 \times 20.3 \times 2.3 = 476.24$$

For the denominator we allow $m$ to range over all values from 1 to 3. This means that the denominator is:

$$A_1F_{21} + A_2F_{22} + A_3F_{23}$$

or

$$\begin{align*}
A_1F_{21} &= 20.1 \times 2.1 \\
A_2F_{22} &= 20.2 \times 2.2 \\
A_3F_{23} &= 20.3 \times 2.3
\end{align*}$$

$(m = 1)$ $(m = 2)$ $(m = 3)$

which is the attractions vector $(20.1, 20.2, 20.3)$ multiplying row 2 of the $F$ matrix $(2.1, 2.2, 2.3)$ term-by-term, and the result added up. So our final result is

$$T_{23} = \frac{10.2 \times 20.3 \times 2.3}{(20.1 \times 2.1) + (20.2 \times 2.2) + (20.3 \times 2.3)} = \frac{476.24}{133.34} = 3.5716$$

Suppose now that we wanted to compute the trips $T_{22}$ made between zones $i = 2$ and $j = 2$. Using the gravity model this would be

$$T_{22} = \frac{O_2A_2F_{22}}{\sum_m A_m F_{2m}}$$

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The useful thing to note here is that the denominator of this expression is exactly the same as the one we’ve just computed. The reason is that this term depends on which origination zone \( i \) we’re concerned with (and this is the same as the previous case, ie \( i = 2 \)), but not on which destination zone. We can see this from the fact that the general formulation of the denominator is \( \sum_m A_m F_{im} \), which depends on \( i \) but not on \( j \). In other words, the denominators for the gravity model expressions for \( T_{21}, T_{22}, T_{23} \ldots \) will be the same, as long as they all involve the same first coordinate (origination zone) \( i \). On the other hand, the denominators for \( T_{12} \) and \( T_{22} \) will be different, because they involve different first (origination zone) coordinates.

That seems straight forward, but there’s one last thing to be aware of. Instead of writing the gravity model as

\[
T_{ij} = \frac{O_i A_j F_{ij}}{\sum_m A_m F_{im}}
\]
as we have done, some authors write it as

\[
T_{ij} = \frac{O_i A_j F_{ij}}{\sum_j A_j F_{ij}}
\]

— that is, using the index \( j \) in the denominator, instead of our \( m \). The important thing to note here is that the two formulations are exactly the same. The \( j \) used in the denominator of the second version is what is called a “dummy index of summation” and has nothing at all to do with the index \( j \) used in \( T_{ij} \). You can think of this as an instruction to begin by computing \( \sum_j A_j F_{ij} = \sum_{j=1}^Z A_j F_{ij} \). One you’ve done this, you’ll have some number \( D \) as a result. Then you can go ahead and compute

\[
T_{ij} = \frac{O_i A_j F_{ij}}{D}.
\]

If you find this confusing, you can always clear things up by re-writing the denominator using an index which does not appear anywhere else in the expression for \( T_{ij} \). Since \( i \) and \( j \) have already been used, we can pick, say, the index \( k \), and write a third equivalent form of the model as

\[
T_{ij} = \frac{O_i A_j F_{ij}}{\sum_k A_k F_{ik}}
\]

which makes it clear, once again, that what’s in the denominator is independent of the \( i \) and \( j \) used in \( T_{ij} \).