Annuities and the IRR

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Annuities

When the temporal impact $\Delta Z_t$ of a project does not vary during some time period, it is said to constitute an *annuity* (for that period).

- So if a project delivers benefits of 10 in each of periods (years) 0 to 15, this is an annuity.
- For an annuity we can write $\Delta Z_t = \Delta Z$ since the impact does not depend in the period we’re talking about.
- Note that a project can define several annuities: for example, costs of $\Delta C$ during the first $t_1$ years, followed by benefits of $\Delta B$ during the next $t_2$ years. This is two separate annuities. What matters is that within each period the impacts do not vary.

Evaluating Annuities (I)

- Obviously if the annuity lasts for only a few years, it’s easy to compute its present value using our present value formula, since there will be only a few terms in the sum.
- But what if it lasts for 30 years? Even with a computer, it would be cumbersome to enter the details.
- Fortunately, there are a couple of formulas that can help.

Evaluating Annuities (II)

- Consider an annuity of $\Delta Z$ which runs from year 0 to year $T$
- The present value of this annuity is

$$ PV = \frac{\Delta Z}{(1+r)^0} + \frac{\Delta Z}{(1+r)^1} + \frac{\Delta Z}{(1+r)^2} + \cdots + \frac{\Delta Z}{(1+r)^T} $$

$$ = \Delta Z \left( s^0 + s^1 + s^2 + \cdots + s^T \right) $$

where $s = 1/(1+r)$ and $r$ is the appropriate discount rate, as we’ve been discussing.
Evaluating Annuities (III)

- We want to evaluate
  \[ PV = \Delta Z \left( s^0 + s^1 + s^2 + \cdots + s^T \right) \]
- Let’s assume \( r > 0 \), so \( 1 < s \), and now focus on the term
  \[ H = \left( s^0 + s^1 + s^2 + \cdots + s^T \right) \]
- The trick is to multiply \( H \) by \( s \), giving
  \[ sH = \left( s^1 + s^2 + \cdots + s^T + s^{T+1} \right) \]
  and then subtract:
  \[ H - sH = s^0 - s^{T+1} \]
  so that
  \[ H(1-s) = s^0 - s^{T+1} \]
  \[ H = \frac{s^0 - s^{T+1}}{1-s}. \]

Evaluating Annuities (IV)

- The result is that the present value of an annuity of \( \Delta Z \) lasting from \( t = 0 \) to \( t = T \) is given by:
  \[ PV = \Delta Z \frac{s^0 - s^{T+1}}{1-s} \]
  \[ = \Delta Z \frac{1 - s^{T+1}}{1-s} \]
where \( s = 1/1 + r \)
- If you imitate this argument, you can see that the present value of an annuity lasting from periods \( t_1 \) to \( T \) is given by:
  \[ PV = \Delta Z \frac{s^{t_1} - s^{T+1}}{1-s} \]

Evaluating Annuities (V)

- Finally, what if the annuity begins in period \( t_1 \) and lasts forever? (This might be true of an environmental improvement, for example).
- We start with the formula
  \[ PV = \Delta Z \frac{s^{t_1} - s^{T+1}}{1-s} \]
- Note that \( s < 1 \) and focus on the term \( s^{T+1} \). As \( T \) gets large we are raising something less than one to successively higher powers. As you can check, the result is that this term gets smaller and smaller. In the limit, it vanishes.
- The upshot is that for an infinite annuity we see that:
  \[ PV = \Delta Z \frac{s^{t_1}}{1-s} \]
where, as always, \( s = 1/1 + r \).

Annuity: First Example

Suppose you’re offered an annuity of 10 in each year from \( t_1 = 1 \) to \( T = 10 \) (inclusive). Suppose that the appropriate discount rate is \( r = 0.02 \). Then \( s = 1/1 + r = 1/1 + 0.02 = 1/1.02 = 0.98039 \) and

\[ PV = 10 \frac{0.98039^1 - 0.98039^{11}}{1-0.98039} \]
\[ = 10 \times 8.9825 \]
\[ = 89.825 \]

Important: remember that the exponent of the second term in the numerator is \( T + 1 \), which here is \( 10 + 1 = 11 \).
**Annuity: Second Example**

- Suppose a project has costs of 10 in years 1–4 and benefits of 5 in years 4–18, so it is composed of two annuities. Suppose the appropriate discount rate is 6%. So $1 + r = 1.06$ and $s = 1/1 + r = 1/1.06 = 0.94340$.
- Annuity 1: amount = $-10$, $t_1 = 1$, $T = 4$ so
  
  $$ PV = -10 \left( \frac{s^1 - s^5}{1 - s} \right) = -10 \left( \frac{0.94340^1 - 0.94340^5}{1 - 0.94340} \right) $$
  $$ = -10 \times 3.4651 = -34.651 $$

- Annuity 2: amount = $5$, $t_1 = 4$, $T = 18$ so
  
  $$ PV = 5 \left( \frac{0.94340^4 - 0.94340^{19}}{1 - 0.94340} \right) $$
  $$ = 5 \times 8.1549 = 40.775 $$

- Conclusion: Project present value is $-34.651 + 40.775 = 6.124$, which is positive, so it looks like a viable project.

**The Internal Rate of Return (I)**

- Consider a stream of impacts $\Delta Z_0, \Delta Z_1, \Delta Z_2, \ldots, \Delta Z_T$ (not necessarily an annuity). The present value of the stream is
  
  $$ PV = \frac{\Delta Z_0}{(1 + r)^0} + \frac{\Delta Z_1}{(1 + r)^1} + \frac{\Delta Z_2}{(1 + r)^2} + \cdots + \frac{\Delta Z_T}{(1 + r)^T} $$

- The Internal Rate of Return (IRR) of the stream is defined as that discount rate $r$ that makes the present value of the stream equal to zero. That is, if we’ve correctly calculated the IRR and found it to be $\tilde{r}$ then, when we plug in and calculate the present value:
  
  $$ PV = \frac{\Delta Z_0}{(1 + \tilde{r})^0} + \frac{\Delta Z_1}{(1 + \tilde{r})^1} + \frac{\Delta Z_2}{(1 + \tilde{r})^2} + \cdots + \frac{\Delta Z_T}{(1 + \tilde{r})^T} $$

  our answer should be zero.

**The Internal Rate of Return (II)**

This raises at least two questions:

- Why would we want to do this?
- How do we actually calculate the IRR?

**Logic of the IRR (I)**

As we have seen, we have several options for the appropriate discount rate. The IRR may allow us to avoid the entire issue. To see this, consider a conventional project with costs up front and benefits in the out-years:

- Suppose we find that the IRR is 20% (0.20); obviously the correct rate (whatever it is) is much less, given today’s economy.
- We know that if we plug $1 + r = 1.20$ into our present value formula formula, the answer will be 0.
- What if we use a rate less than 20%? At any smaller rate, all the impacts will be discounted less, and hence will grow in present value.
- So the present value of the entire project must be positive.
- And this holds for any rate less than our IRR.

We conclude that in a sense the rate doesn’t matter: at any reasonable rate, the project will have a positive present value.
Logic of the IRR (II)

- Conversely, suppose that the IRR is tiny, something like $0.1 \times 10^{-5} = 0.00001$.
- We know that this is much too low (even today).
- What would happen if we used a larger — and more correct — rate?
- The out-year impacts would be discounted more, hence their contribution to the stream’s present value would fall.
- This would make the present value of the stream negative.

We conclude that at any reasonable discount rate, the project’s present value is negative: it is a guaranteed loser.

Once again, if we know the IRR, the “correct” discount rate turns out not to matter.

Calculating the IRR

Now that we know why we might want to calculate the IRR, how do we do so? Unfortunately, this is one of those cases where there’s usually no other option but trial and error. The steps are:

1. Pick a discount rate ($r$) and calculate the project’s present value.
2. If the result is positive, you want to revise your guess for $r$ to make the result smaller (closer to zero). You do this by increasing the discount rate.
3. If the result is negative, you want to revise your guess for $r$ to make the result larger (closer to zero). You do this by decreasing the discount factor.
4. Repeat until the present value is close enough to zero.

Calculating the IRR — Example (I)

- Suppose our stream of impacts, in years 0 to 3, is $-5, -3, 6, 7$.
- What’s the IRR?
- To make things easy, try $r = 0$ as a first guess. In this case the present value is the unweighted (undiscounted) sum of the impacts, i.e. $PV = 5$.
- Since this is too large, we want to try with a larger discount rate in order to reduce the present value.
- Try $r = 0.10$. Then $PV = 2.49$. This is smaller than 5 (from our first guess), so we’re moving in the right direction; but not good enough. To make it smaller still, increase $r$ again.
- Try $r = 0.25$. Then $PV = 0.024$. We want to increase $r$ again.

Calculating the IRR — Example (II)

Continuing, we find:

- If $r = 0.27$, $PV = -0.225$. So now we want to make $r$ a little smaller.
- If $r = 0.26$, $PV = -0.102$. If $r = 0.255$, $PV = -0.0396$. If $r = 0.252$, $PV = -0.0015$.
- If $r = 0.2519$, then $PV = 0.0003$, which is probably close enough to zero.

We conclude that the IRR for this project is about 25.19%. Note that by our previous logic this means that our project seems to be a winner, independently of the “correct” value of the appropriate discount rate (given today’s rates).
So far, so good. But there are some problems with the IRR.

**Problem 1:** The logic of the IRR relies on the computed IRR being either unreasonably large or unreasonably small.

- But what if the IRR turns out to be a *reasonable* value: today, perhaps something like 2%. This isn’t *obviously* too large or too small.
- In this case the logic of the IRR doesn’t help us at all.

This is a quadratic equation. So in general there will be two solutions. They might not both be economically meaningful (i.e. real, and non-negative) but then again, they might be.

- What about a 4-impact project? The IRR solves the equation

\[ 0 = \Delta Z_0 + s(\Delta Z_1) + s^2(\Delta Z_2) + s^3(\Delta Z_3) \]

for \( s \). This may have as many as three solutions. And so on.

- So for any realistic project, we’re likely to have several, perhaps tens, of candidates for an IRR. Which to choose? The theory doesn’t say.

For all these reasons, the IRR is nowadays rarely used in practical project evaluation. If you want to see how your project turns out, it is better to sensitivity-test your present-value computations with upper and lower bounds on the appropriate discount rate, as we’ve previously discussed.

Interestingly enough, the IRR is used in practical finance. The yield on a security as printed in the newspapers or available online, is the solution to an IRR problem.