Consumer’s Surplus Calculations

Philip A. Viton

November 4, 2014
We have seen that the change in consumer’s surplus

$$\Delta CS = CS(p_1^b) - CS(p_1^a)$$

represents an empirically implementable measure of the consumer’s willingness-to-pay for a project that changes the own-price of a market good $x_1$ from $p_1^a$ to $p_1^b$.

The question is, how do we use it?

We assume that we have estimated the (community or individual) demand function for the good whose price is changed by our project (here, $x_1$).
The Linear-in-Own-Price Case

- We begin with the easiest, and probably most common, case: where the demand function is linear in own-price. (For simplicity we’ll refer to this as simply the linear case).
- This is what you will most often see when you run a regression of quantity demanded on own-price and other determinants of demand.
- The important thing here is that although the own-price must enter the demand function linearly, any other variables can enter non-linearly.
- So the following are linear in own price (using the obvious subscript notation):

  \[ x^* = \alpha + \beta p_x \]
  \[ x^* = \alpha + \beta p_x + \gamma e^{py} \]
  \[ z^* = \alpha + \beta p_z p_x + \gamma p_x^2 \]

- And the following are not:

  \[ x^* = \alpha + \beta p_x + \gamma p_x^2 + p_y \]
  \[ z^* = \alpha + \beta p_z + \gamma p_z^2 + \delta M \]
In the figure, the demand function for $x_1$ is linear in own-price.

The change in consumer’s surplus is the signed area under the demand function between the two price horizontals.

In this case, the required area is the area of a rectangle plus the area of a triangle.
This suggests an easy strategy:

- Take $p_1^a$, plug it into the demand function, and compute $x_1^a$.
- Take $p_1^b$, plug it into the demand function, and compute $x_1^b$.
- This gives us all the information we need to compute the area, recalling that
  
  - area of a rectangle $= (\text{base length}) \times (\text{height})$
  - area of a triangle $= (\text{half the base length}) \times (\text{height})$

- But remember that this way of computing $\Delta CS$ is valid only when demand is linear-in-own-price.
Warning!!

If you compute the change in consumer’s surplus as an area you need to remember to get the sign right, since areas are by definition positive.

- If the project reduces the price of a good, then wtp is positive and consumer’s surplus is positive.
- If the project increases a price, then wtp is negative and the change in consumer’s surplus is negative.
- So at the end of your area computation you must make sure that the sign is correct.
- If you forget this, and your project raises the price you will be making a serious error: your conclusion will be that wtp is positive when it should be negative.
Linear Demand: Example 1 (I)

Suppose the demand for $x_1$ is:

$$x_1^*(p_1): \quad x_1 = 18 - \frac{1}{2} p_1$$

which is obviously linear in $p_1$; and suppose the project reduces $p_1$ from $p_1^a = $10 to $p_1^b = $8.5.

We now implement our strategy for computing $\Delta CS$ as an area.
Plug in $p_1^a$ and compute $x_1^a$:

$$x_1^a = 18 - \frac{1}{2} p_1^a$$

$$= 18 - \frac{1}{2} \times 10 = 18 - 5 = 13$$

Plug in $p_1^b$ and compute $x_1^b$:

$$x_1^b = 18 - \frac{1}{2} p_1^b$$

$$= 18 - \frac{1}{2} \times 8.5 = 18 - 4.25 = 13.75$$
Then the area of the rectangle is:

\[(10 - 8.5) \times 13 = 19.5.\]

The area of the triangle is:

\[\frac{1}{2}(13.75 - 13)(10 - 8.5) = \frac{1}{2}(0.75)(1.5) = 0.5625.\]

So the total area is \(19.5 + 0.5625 = 20.063.\)

Since this is a price reduction, the sign is positive and we have:

\[\Delta CS = 20.063\]
Suppose the demand for $x_1$ is:

$$x_1^*(p_1) : \quad x_1 = 18 - \frac{1}{2} p_1$$

and suppose the project increases $p_1$ from $p_1^a = $8.50 to $p_1^b = $10.

This is the same as Example 1 except that the initial and final prices are interchanged, so we have a price increase for good 1.

All the area calculations remain valid, ie are exactly the same as in the previous example.

All we need to do is to remember to get the sign correct. Since this is a price increase the sign is negative and we have:

$$\Delta CS = -20.063$$
What about the units of our results? (20.063 what??)

- It depends on the units of the demand function. If price is in dollars and quantity is measured in single units, then the result is in dollars. If price were in cents, and quantity in single units, then the result would be in cents. If the price were in cents and quantities were measured in hundreds of units (so that 9.4 would stand for 9400 units) then the result would be in hundreds of cents.
- It also depends on the time units of demand (units demanded per day, or per year, etc).
- Remember that $\Delta CS$ is in effect an area, so the same rules apply here as to any area computation.
Linear Demand : Example 3 (I)

Suppose we have a more complicated demand function:

$$x^*_1(p_1, p_2) : \quad x_1 = 16 - \frac{1}{2}p_1 - 0.1p_2^2$$

where $p_2$ could be any terms, linear or nonlinear, as long as they do not involve the own-price, $p_1$. Note that in this case the term involving $p_2$ is quadratic (ie not linear) in $p_2$. However, demand is linear in $p_1$, which is all we need here.

Suppose now that the project increases $p_1$ from $p^a_1 = 8.5$ to $p^b_1 = 10$.

The important point here is that in order to get started we need to know the fixed (and assumed unchanging) values of all the other terms in the demand function, in this case $p_2$. Let's suppose that $p_2 = 4$. 
To calculate $\Delta CS$ for this project:

- First plug in the values of all the unvarying data (ie, everything except the own-price) and find:

  \[ x_1 = 16 - \frac{1}{2} p_1 - (0.1 \cdot 4^2) \]
  \[ = 16 - \frac{1}{2} p_1 - 1.6 \]
  \[ = 14.4 - \frac{1}{2} p_1 \]

The result is that we now have a linear-in-$p_1$ demand function and we can proceed as before.
Plug in $p_1^a$ and compute $x_1^a$:

$$x_1^a = 14.4 - (0.5 \times 8.5) = 10.15.$$ 

Plug in $p_1^b$ and compute $x_1^b$:

$$x_1^a = 14.4 - (0.5 \times 10) = 9.4.$$
Area of rectangle:

\[(10 - 8.5) \times (9.4) = 14.1.\]

Area of triangle:

\[\frac{1}{2} (10 - 8.5)(10.15 - 9.4) = 0.5625.\]

Total area = 14.1 + 0.5625 = 14.6625.

Sign? This is a price increase, so the sign is negative, and we conclude that

\[\Delta CS = -14.6625\]

(per unit time of demand).
If the demand function isn’t linear in own-price, then our geometrical computations no longer work.

The difficulty is that the right-hand part of the shaded area is no longer exactly a triangle.

What can we do?
Fortunately, the mathematicians have solved this problem for us, and all we need to do is apply their solution.

But before we do, we need to develop a shorthand.

For the change in consumers surplus, we want to find the area under a demand function when the own-price changes from an initial (pre-project) value to a final (post-project) value.

And we’ve seen that the demand functions can depend on lots of other variables besides the own-price, so long as we consider them as fixed (as with $p_2$ in Example 3).

So we want to find the area under the demand function for some good, which depends on the own-price and on other variables, when the own-price changes from pre-project to post-project values.
Mathematical Setup (II)

- Let’s forget for the moment about the economic content of our problem.
- The underlying task is to find the area under a curve described by a function $f(u, v)$ when $v$ is fixed, and $u$ varies from $u = a$ to $u = b$.
- The link to the wtp problem is that $f$ is the demand function, $u$ is the own-price, $v$ is all other determinants of demand, and $a$ is the pre-project (initial) price and $b$ is the final (post-project) price.
A New Notation

- The mathematicians symbolize the phrase “Area under $f(u, v)$ when $u$ varies between $u = a$ and $u = b$, and $v$ is fixed” as:

\[
\int_{u=a}^{u=b} f(u, v) \, du
\]

or, slightly compressed:

\[
\int_{a}^{b} f(u, v) \, du
\]

- The symbol $\int$ is called an integral symbol, and the whole expression is called the (definite) integral of $f$.

- The $du$ at the end of this is to remind us that it is $u$ (and not $v$) that is varying. (There could be circumstances when we might want to know something about the area under $f$ when $v$ varies, in which case we’d write $dv$ instead of $du$).
Computing Integrals (I)

Of course, what we really need to know is how to compute these integrals (areas). This turns out to be quite straightforward. Here’s the sequence of steps. To compute the value of

\[
\int_a^b f(u, v) \, du
\]

1. Plug in the given values for \(v\) (note that \(v\) can be a list of other factors that determine demand). The result is a form of \(f\) that depends only on \(u\): \(f(u)\).
2. Find a special function \(F(u)\) that is related to \(f(u)\). This function is called an anti-derivative of \(f(u)\).
3. Plug \(b\) into \(F\) and compute \(F(b)\)
4. Plug \(a\) into \(F\) and compute \(F(a)\)
5. The area we want is given by:

\[
\text{Area} = F(b) - F(a)
\]
Obviously, the only step that requires any thought is step 2 where we need to find the anti-derivative $F(u)$. How do we do this?

The answer is that we look it up, either in a calculus textbook (you’ll often find tables of anti-derivatives on the inside covers of these texts) or in a lookup table.

A separate class handout provides a lookup table that covers the forms that you will most likely run into in practice.

In an examination, a table like the handout table will be provided for you. You do not need to memorize this table (though after you’ve done a few examples, you may find that you don’t need to look up the answers in the table as often as you once did).
Integrals and Consumer’s Surplus

The relation between an integral and the change in consumer’s surplus is as follows:

- Suppose our project changes the own-price of a good $x_1$ from $p_1^a$ to $p_1^b$.
- And suppose that $x_1$ has a demand function $f(p_1, v)$, where $v$ stands for all determinants of demand other than the own-price.
- Then

\[
\Delta CS = - \int_{p_1^a}^{p_1^b} f(p_1, v) \, dp_1
\]

\[
= - ( F(p_1^b) - F(p_1^a) )
\]

where $F$ is an anti-derivative corresponding to $f$. In other words $\Delta CS$ is *minus the integral* or *minus the area*. 
Careful!!

- **Note the minus sign!!** This is what saves you the trouble of checking the sign of your answer. If the project reduces $p_1$ (so $p_1^a > p_1^b$) then the integral-based computation will result in $\Delta CS$ being positive, and if the project increases $p_1$ then $\Delta CS$ will be negative, just as we want.

- It is easy to forget the minus sign. So when you do any computations, it is worth giving them a sanity check: if the project raises prices and you’ve computed a positive $\Delta CS$ then you know that something has gone wrong, and you should check whether you’ve forgotten the minus sign.
Let’s re-do Example 3 using our new methods. Of course, Example 3 was a case of linear-in-own-price demand, so we could do it geometrically. But to gain confidence in our new method, it’s useful to do a computation where we know the correct answer in advance.

For Example 3 we had

\[ x_1 = 16 - \frac{1}{2} p_1 - 0.1 p_2^2 \]

with \( p_2 = 4, \ p_1^a = 8.5, \ p_1^b = 10. \)

We therefore want to compute

\[ \Delta CS = - \int_{8.5}^{10} \left( 16 - \frac{1}{2} p_1 - 0.1 p_2^2 \right) dp_1. \]

We shall do this in the order indicated above.
We begin by simplifying the demand function, so we plug in values for all the variables except $p_1$. In this case the only such variable is $p_2 = 4$ so we have:

$$f(p_1) = 16 - \frac{1}{2} p_1 - 0.1 p_2^2$$
$$= 16 - \frac{1}{2} p_1 - 0.1 \cdot 4^2$$
$$= 16 - \frac{1}{2} p_1 - 1.6$$
$$= 14.4 - \frac{1}{2} p$$

(as we saw before).
2. Our next task is to find the anti-derivative of $f(x) = 14.4 - \frac{1}{2}p$. We do this term-by-term.

- Using Rule 1 with $c = 14.4$, the anti-derivative of 14.4 is $14.4p_1$.
- Using Rule 2 with $c = \frac{1}{2}$ the anti-derivative of $\frac{1}{2}p_1$ is $\frac{1}{2} \left( \frac{p_1}{2} \right)^2 = \frac{1}{4}p_1^2$.
- So the anti-derivative of the entire expression is

$$F(p_1) = 14.4p_1 - \frac{1}{4}p_1^2$$
Example 4 — Details (III)

From this point on, it’s just a matter of routine:

3. Plug in $p_1^b = 10$ and compute $F(p_1^b)$:

$$F(p_1^b) = F(10) = (14.4 \times 10) - \left( \frac{1}{4} \times 10^2 \right) = 119.0$$

4. Plug in $p_1^a = 8.5$ and compute $F(p_1^a)$:

$$F(p_1^a) = F(8.5) = (14.4 \times 8.5) - \left( \frac{1}{4} \times 8.5^2 \right) = 104.3375$$
5. Finally, our answer is given by:

\[ \Delta CS = -( F(p_1^b) - F(p_1^a) ) \]
\[ = -(119.0 - 104.3375) \]
\[ = -14.6625 \]

which agrees with Example 3.
Example 5 — Setup

Let’s do an example where geometry can’t help us.

- Suppose the demand for a good $z$ is:

$$z = 16 - \frac{1}{2} p_z + \frac{1}{10} p_z^2 - p_y$$

and suppose $p^a_z = 10$ (pre-project price) and $p^b_z = 9$ (post-project price); and that $p_y = 4$.

- We want to compute the change in consumer’s surplus as minus the area under this demand function when the own-price $p_z$ varies from $p^a_z$ to $p^b_z$, that is:

$$\Delta CS = - \int_{10}^{9} (16 - \frac{1}{2} p_z + \frac{1}{10} p_z^2 - p_y) \, dp_z$$

- Note that because of the term $\frac{1}{10} p_z^2$, the demand function is nonlinear in $p_z$, so geometry won’t work.
Example 5 — Details (I)

We work through our steps in order.

1. Simplify. Plug in $p_y = 4$ and find:

\[
z = 16 - \frac{1}{2}p_z + \frac{1}{10}p_z^2 - p_y
= 16 - \frac{1}{2}p_z + \frac{1}{10}p_z^2 - 4
= 12 - \frac{1}{2}p_z + \frac{1}{10}p_z^2
\]
2. Find the anti-derivative:

- From Rule 1 with $c = 12$ the anti-derivative of 12 is $12p_z$
- From Rule 2 with $c = \frac{1}{2}$ the anti-derivative of $\frac{1}{2}p_z$ is $\frac{1}{2}p_z^2 = \frac{1}{4}p_z^2$
- From Rule 3 with $c = \frac{1}{10}$ or Rule 4 with $c = \frac{1}{10}$ and $d = 2$, the anti-derivative of $\frac{1}{10}p_z^2$ is:
  \[
  \frac{1}{10} p_z^3 = \frac{1}{30} p_z^3
  \]

- So, re-assembling, the anti-derivative of the entire expression is:
  \[
  F(p_z) = 12p_z - \frac{1}{4}p_z^2 + \frac{1}{30}p_z^3
  \]
Example 4 — Details (III)

3. Plug in $p_z^b = 9$ and find:

$$F(p_z^b) = (12 \times 9) - \left( \frac{1}{4} \times 9^2 \right) + \left( \frac{1}{30} \times 9^3 \right)$$
$$= 112.05$$

4. Plug in $p_z^a = 10$ and find:

$$F(p_z^a) = (12 \times 10) - \left( \frac{1}{4} \times 10^2 \right) + \left( \frac{1}{30} \times 10^3 \right)$$
$$= 128.333\overline{3}$$

5. So:

$$\Delta CS = -(F(p_z^b) - F(p_z^a))$$
$$= -(112.05 - 128.333\overline{3})$$
$$= 16.283\overline{3}$$

Note that we get the sign right without having to adjust it manually.
Final Remark

- The course website has a handout that contains a set of practice examples on integrals and the change in consumer’s surplus. (It is in the Other Class Handouts section).
- If this material is new to you, you may want to try your hand at some of the examples, for practice.
Appendices
If you remember your calculus, you’ll remember that our statement that the anti-derivative of \( f(u) \) is \( F(u) \) is slightly inaccurate. The correct statement is that the anti-derivative of \( f(u) \) is \( F(u) + c \) where \( c \) is an arbitrary constant. In some cases the constant can be an interesting and important part of the analysis.

But not when we want to compute \( \Delta CS \). The reason is that we’re not directly interested in the anti-derivative \( F \) by itself, but only in

\[
\Delta CS = -(F(u_1) - F(u_2))
\]

that is, in differences in \( F \)’s. And when we compute that, the \( c \)’s drop out. So you don’t need to worry about the arbitrary constant.
Appendix 2: Calculus-Based Check

- If you remember your calculus, you have an easy way to check that you’ve computed the correct anti-derivative.
- If $F(u)$ is really an anti-derivative of $f(u)$ then $dF(u)/du = f(u)$: that is, if you differentiate your anti-derivative, you should get back the original function.
- For instance, in Example 5, we found that the anti-derivative was:

$$F(p_z) = 12p_z - \frac{1}{4}p_z^2 + \frac{1}{30}p_z^3$$

If we differentiate this with respect to $p_z$ we get:

$$12 - \frac{1}{2}p_z + \frac{1}{10}p_z^2$$

which is the demand function after we have plugged in $p_y = 4$. So our anti-derivative is correct.
Appendix 3 : Using the Lookup Tables (I)

- The lookup table distributed in class tells you how to find an anti-derivative of an expression $f(u)$; but when we use it for consumer’s surplus calculations we’re interested in expressions like $f(p_1)$ or $f(p_x)$. etc. What’s the relationship?
- First of all, it’s useful to understand that in the expression

$$\int f(u)\, du$$

the variable $u$ is what is called a “dummy variable” of integration. That is, you can replace $u$ by any unused variable you like, provided you do it consistently. This means that the following expressions are all the same:

$$\int f(u)\, du \quad \int f(x)\, dx \quad \int f(p_1)\, dp_1 \quad \int f(p_x)\, dp_x$$
And this means that when you need to compute an anti-derivative for $f$ in an expression like

$$\int f(p_1) \, dp_1$$

what you need to do is look in your table for an expression matching $f$ in which $p_1$ occupies the same position in your expression as $u$ does in the table.

So for instance if you’re looking for an anti-derivative for $\beta_1 p_1^2$, you’re looking for an entry matching $\beta_1 u^2$, or, recognizing that $\beta_1$ is a constant, one matching $cu^2$. 