Identifying the Discount Rate

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We continue our study of time preferences and individual wtp for the impacts of time-dependent project impacts:

- We study a single utility-maximizing individual in a 2-good (consumption, measured in dollars), 2-period world.
- No inflation or technical change.
- We have seen that under these assumptions the present value of (wtp now for) $\Delta Z_t$ received in period $t$ is:

$$\text{PV}(\Delta Z_t) = \frac{\Delta Z_t}{(1 + \rho)^t}$$

where $-(1 + \rho)$ is the common slope of the individual’s 1-period intertemporal indifference curve at the observed bundle.
Our problem is: how to identify the slope?

In fact we have two problems:

1. which indifference curve will our individual be on?
2. where on that indifference curve will she be (which intertemporal consumption bundle will she choose)?

Both of these can affect the slope of the indifference curve, especially if the curves are not radial blow-ups of one another.
So far we have examined only the individual’s preferences.

We now ask: what will the individual do? Where on her (intertemporal) indifference curve will she choose to be?

This will determine the slope of the indifference curve at the individual’s pre-project position.

This involves a consideration of not just her preferences but her opportunities to satisfy those preferences.
Income and Consumption (I)

- We suppose that our individual has income $M_0$ and $M_1$ in the two periods.
- We assume that she is in an economy with a banking system.
- The banks have an interest rate of $r$ per annum. To simplify our analysis for now, we suppose that the individual can both borrow and lend at the same rate $r$ (the bank rate).
- The important consequence of this is that the individual’s consumption ($Z_0$ and $Z_1$) in each period can differ from her income in that period.
  - In period-0, she can consume more than her period-0 income by borrowing against period 1 income.
  - In period-1, she can consume more than her period-1 income by saving in period 0.
Let's develop an equation relating our individual’s income and consumption through time: this will be her *intertemporal budget constraint*.

Suppose she is a period-0 *saver*: she consumes less in period 0 than her period-0 income. Her period-0 savings are $M_0 - Z_0$.

Given the banking system, she can lend this to the bank (deposit it). After 1 year she gets back her deposit (savings) plus interest at rate $r$, so she has $(M_0 - Z_0) + r(M_0 - Z_0) = (M_0 - Z_0)(1 + r)$.

In addition, she has her period-1 income ($M_1$).

So her opportunity for period-1 consumption $Z_1$ satisfies:

$$Z_1 = M_1 + (M_0 - Z_0)(1 + r).$$

This is her intertemporal budget constraint.
To plot the budget constraint, we need to identify 2 points on it.

As usual, we find points on the two consumption axes.

If she consumes everything in period 0, we identify the point \((Z_0, 0)\), with zero consumption in period 1. So set \(Z_1 = 0\) and solve for \(Z_0\):

\[
\begin{align*}
0 & = M_1 + (M_0 - Z_0)(1 + r) \\
Z_0(1 + r) & = M_1 + M_0(1 + r) \\
Z_0 & = \frac{M_1}{1 + r} + M_0
\end{align*}
\]

Conversely if she consumes nothing in period 0 \((Z_0 = 0)\) her period-1 consumption (point \((0, Z_1)\)) will be:

\[
\begin{align*}
Z_1 & = M_1 + (M_0 - 0)(1 + r) \\
& = M_1 + M_0(1 + r)
\end{align*}
\]
Intertemporal Budget Constraint

The slope of the intertemporal budget constraint is:

\[
\frac{Z_1 - 0}{0 - Z_0} = -\frac{Z_1}{Z_0}
\]

\[
= -\frac{M_1 + M_0(1 + r)}{(1+r) + M_0}
\]

\[
= -\frac{M_1 + M_0(1 + r)}{M_1+M_0(1+r)}
\]

\[
= -(1 + r)
\]
The individual will select her intertemporal consumption pattern to maximize utility subject to her budget constraint. The condition for maximization is that the budget constraint be tangent to an indifference curve.
In an equilibrium defined by the tangency condition, the slope of the budget constraint will equal the slope of the intertemporal indifference curve.

So at the equilibrium point we will have:

\[
\text{slope of indiff curve} = \text{slope of budget constraint} \quad -(1 + \rho) = -(1 + r).
\]

We conclude that for an individual in utility-maximizing equilibrium

\[
\rho = r.
\]
So we now have a way of empirically identifying her $\rho$: just look up the applicable bank rate $r$.

Note that this logic will apply to *any* individual: we don’t need to know anything about her indifference curves or her income, as long as there’s a banking system.

She will do all the work for us: her utility maximization subject to her intertemporal budget constraint will cause her to adjust her behavior (consumption today versus consumption tomorrow) until she achieves a tangency between the indifference curve and the budget constraint.
So our result is:

- Suppose a project delivers $\Delta Z_t$ in period $t$. An empirical estimate of her period-0 wtp — her present value — for this is:

$$PV(\Delta Z_t) = \frac{\Delta Z_t}{(1 + r)^t}$$

where $r$ is the bank rate.
We can now pull all this together.

Suppose a project delivers small impacts $\Delta Z_0$, $\Delta Z_1$, $\Delta Z_2$ \ldots $\Delta Z_T$ to this individual in periods $0, 1, 2, \ldots, T$. (Note that some of these can be negative, i.e., costs).

We replace each of these by its present value (year-0 equivalent).

We can now consistently add up the present values, since they are all period-0 quantities.
Then the present value (wtp) of the entire project is:

\[ PV = \frac{\Delta Z_0}{(1 + r)^0} + \frac{\Delta Z_1}{(1 + r)^1} + \frac{\Delta Z_2}{(1 + r)^2} + \cdots + \frac{\Delta Z_T}{(1 + r)^T} \]

Which we can write compactly as:

\[ PV = \sum_{t=0}^{T} \frac{\Delta Z_t}{(1 + r)^t} \]
Large and Small Projects (I)

Let’s remind ourselves why this only works for small projects.

Basically, the reason is that we are measuring willingness-to-pay along the budget constraint \( \text{slope} = -(1 + r) \) instead of along the indifference curve \( \text{slope} = -(1 + \rho) \).

The reason we are doing this, of course, is that we don’t know her indifference curves.

When the project keeps us near the initial (equilibrium) point, the two will be close; but as we move further away, they may not be.
Starting from an equilibrium at A, consider a large project $\Delta Z'_1$:

- Along the budget constraint, we would estimate the WTP today as $\Delta Z'_0$
- This is quite inaccurate. The correct answer (using the indifference curve) is $\Delta Z''_0$. 
Starting from an equilibrium at A, consider a small project $\Delta Z_1$:

- Along the budget constraint, we would estimate the wtp today as $\Delta Z_0$
- This is pretty close to the correct answer along the indifference curve.
We now need to deal with three problems that we have so far ignored:

1. Different bank rates for borrowing and lending.
2. The problem of inflation.
3. The problem of technical change.
There are at least two bank rates that everyone faces: the rate the bank charges you to borrow, and the rate it pays you on your deposits (savings).

Which should we use?

Typically, a public project asks you (the taxpayer) to give up something today in exchange for a benefit tomorrow.

This amounts to asking you to be a period-0 saver.

Therefore, in the evaluation of public projects, the relevant individual rate is the rate the bank pays on your savings, and not the rate it charges you to borrow from it.
Let’s begin with our 2-period no-inflation world.

Suppose that in order to compensate for a small change (sacrifice) of \( \Delta Z_0 \) dollars today we require \( \Delta Z_1 \) dollars tomorrow. So the slope of the intertemporal indifference curve is \( \Delta Z_1 / \Delta Z_0 = -(1 + \rho) \) (as before).

Now suppose we have a general inflation at rate \( (1 + i) \) per year.

With inflation, in order to persuade you to give up \( \Delta Z_0 \) dollars today we would have to give you \( (1 + i)\Delta Z_1 \) dollars tomorrow because, with inflation, \( \Delta Z_1 \) will purchase less than it did before.

So with inflation the slope of the intertemporal indifference curve increases to \( -(1 + \rho)(1 + i) \).
But what about the budget constraint? If there is a *general* inflation, the bank rate — which is a price just like any other price — will also increase, to $(1 + i)(1 + r)$.

So the slope of the intertemporal budget constraint will be $-(1 + i)(1 + r)$.

Then in equilibrium we will have:

\[
\text{slope of indiff curve} = \text{slope of budget constraint}
\]

\[
-(1 + \rho)(1 + i) = -(1 + i)(1 + r)
\]

so we still have:

\[
\rho = r.
\]
But we need to be careful: $\rho$ is the no-inflation pure time preference rate, and $r$ is the no-inflation bank rate.

The corresponding inflated quantities are $(1 + \rho)(1 + i)$ (time preference) and $(1 + i)(1 + r)$ (bank rate).
Inflation (IV)

This means that when we do project evaluation we have a choice:

1. We can do the entire analysis in real (the US Commerce Department now refers to this as chained) terms: that is, with all quantities expressed in the purchasing power of the starting time period. This is the no-inflation way to do things and in some ways it is easier, since it doesn’t require you to forecast inflation. When we do the analysis this way, we want to use a non-inflated bank rate.

2. We can do the entire analysis in nominal (the US Commerce Department refers to this as current) terms: that is in monetary units whose values increase over time partly due to inflation, and not necessarily because we are providing actual benefits to people. When we do the analysis this way, we want to use a bank rate that has inflation built in.
How do we tell whether a quantity is real or nominal?

- One way is to imagine that inflation suddenly picks up significantly: would you expect the quantity to change in response to this? If you do then the quantity is nominal; otherwise it is real.
- This means, for example, that the interest rates quoted by the local banks are all nominal rates.
If you have a nominal rate, one quick way to convert it (approximately) to a real one is to subtract the rate of inflation: a nominal rate of 5% in the face of annual inflation of 3% is roughly equivalent to a real rate of 2%. (See Appendix).

For the United States, there is no need to compute a real rate from a nominal one: the US Treasury issues inflation-adjusted bonds, whose return changes automatically with the rate of inflation so as to maintain its purchasing power. This becomes a hedge against inflation for savers. We can use the yield on these bonds as our real interest rates.

Other countries issuing inflation-adjusted securities include the UK, France, Germany, Japan, Hong Kong, Italy, and Greece.
Enter The Planner (I)

So far we have taken a completely individualistic approach to discounting: an individual is offered a benefit tomorrow, and we have asked what sacrifice (wtp) he or she is willing to make today in order to get it.

But there are a number of reasons why one might wish to take a broader (non-individualistic) view:

- Technical change: in a world in which productivity is improving over time, consumption opportunities will go up, irrespective of whether we invest today.

- Population growth: if the population is increasing and we do not alter our investment policy today, then (other things being equal) future generations will be worse off than we were.

It is not clear that an individual will take these into account when adjusting her behavior in equilibrium. But doing so is a task for the planner.
Enter the Planner (II)

We will assume:

- The planner faces a world of identical individuals, so that he or she can do the analysis in terms of a representative (average) individual.
- The goal is to maximize the discounted sum of the representative individual’s utilities, i.e., the sum of the present values of the individual’s stream of utility over time.
- At any time, available resources can be used for consumption or investment; the amount invested influences future consumption possibilities.
The Social Discount Rate (I)

In a model that takes these into account, it can be shown that future impacts should be discounted using the *social discount rate* $r^*$ which is given by the formula:

$$r^* = \rho + \eta \theta - \nu$$

where:

- $\rho$ is the pure (no-inflation, no-technical-change) time-preference rate (as before).
- $\eta > 0$ is the *elasticity* of the marginal utility of consumption, i.e. the percentage change in the marginal utility of consumption per unit percentage change in utility.
- $\theta$ is the rate of technical change.
- $\nu$ is the rate of population change.
The Social Discount Rate (II)

In the formula for the social discount rate:

\[ r^* = \rho + \eta \theta - \nu \]

note that the individual terms make intuitive sense:

- If there is no technical change \((\theta = 0)\) and no population change \((\nu = 0)\) then \(r^* = \rho\) as before
- If there is just technical change, then, everything else equal, future generations will be better off than we are. Thus, when we evaluate projects, we want to give less weight to future benefits. This implies that we should discount the impacts using a rate that is higher than \(\rho\), which is what the plus sign in the formula implies.
- If there is only population growth then, everything else equal, future generations will be worse off than we are. To remedy this, we want to give more weight to future benefits. This implies that we should use rate that it lower than \(\rho\), which is what the minus sign implies.
The Social Opportunity Cost of Capital (I)

There is another approach to identifying the appropriate discount rate, known as the social opportunity cost of capital. Its logic is:

- Suppose there is a fixed amount of investment capital available.
- If the government uses capital (on our project) then it is not available to the private sector.
- The private sector’s return on the funds therefore constitutes the next-best use of the capital.
- This is by definition the (social) opportunity cost of capital.
- Note that the basic assumption — a fixed investment pool — may not be correct: think about China, which at one stage seemed to be supplying as much capital as needed, so the investment pool for the US was not fixed (though recently that has become a bit more doubtful).
We have now developed several choices for a discount rate. Let’s try to estimate them, to see how (if) they differ. We have two strategies here: to base estimates on past averages, or to look at projections of the future. Neither approach is foolproof, and projections, especially for the longer-term, tend to be somewhat unreliable (but on the other hand it’s not clear that using the past as guidance is any more reliable). Nevertheless, we’ll take the projections route. See Data Appendix for more on sources.
In some cases it may be possible to avoid all data-munging.

For Federal projects there are OMB rules as to which discount rate to use.

For these projects, the analyst has no discretion. Here are OMB’s current rates:

<table>
<thead>
<tr>
<th>Project Length:</th>
<th>5yr</th>
<th>10yr</th>
<th>20yr</th>
<th>30yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real rate</td>
<td>0.0%</td>
<td>1.0%</td>
<td>1.6%</td>
<td>1.9%</td>
</tr>
<tr>
<td>Nominal rate</td>
<td>1.9%</td>
<td>3.0%</td>
<td>3.6%</td>
<td>3.9%</td>
</tr>
</tbody>
</table>

Even if you have to use these, there’s an interesting question as to how the other possible rates compare to them: OMB could simply be wrong.
Estimates: Personal Discount Rates

- We can use US Treasury long-term bond rates as proxies for individual investment opportunities, and hence for the slopes of their intertemporal budget constraints.

- Note that these are generally assumed to be risk-free rates: they are based on the assumption that no matter what, a bond backed by the “full faith and credit” of the Unites States will be repaid.

- In the table below, the real rate is based on Treasury inflation-adjusted securities (called TIPS).

<table>
<thead>
<tr>
<th>Project Length:</th>
<th>5yr</th>
<th>10yr</th>
<th>20yr</th>
<th>30yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real rate</td>
<td>0.0%</td>
<td>0.3%</td>
<td>0.6%</td>
<td>0.8%</td>
</tr>
<tr>
<td>Nominal rate</td>
<td>1.3%</td>
<td>2.1%</td>
<td>NA</td>
<td>2.9%</td>
</tr>
</tbody>
</table>
Next, let’s estimate the social discount rate.

- We already know that we can identify $\rho$ by $r$, the bank rate (for a utility maximizing consumer).
- We will estimate the rate of technical change, $\theta$, as the annual rate of change in real personal consumption expenditures per capita. (We want this to be a real quantity since if inflation picks up there’s no reason to believe that the rate of technical change — invention, progress — will alter).
- There’s a question of how to measure population growth ($\nu$). We could use the population, as the name suggests, but some authors, suggest we should use the growth rate in the labor force, since the issue is productivity. For our estimates we’ll use population.
The serious difficulty is going to be the marginal utility of consumption (and its elasticity): how exactly does an increase in consumption contribute to an increase in utility?

Zhuang, Liang, Lin and Guzman (2007, table 2) survey some estimates of the elasticity of the marginal utility of income. If we look at their category “indirect behavioral evidence”, their estimates range from 1.06 to 1.97 (mostly UK). Let’s take the average, about 1.50. We’ll use this for $\eta$. 
Based on this, here are estimates of the real social discount rate. In this, $\rho$ is a real rate.

<table>
<thead>
<tr>
<th>Project Length</th>
<th>$\rho$</th>
<th>$\eta \cdot \theta$</th>
<th>$- \nu$</th>
<th>$r^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5yr</td>
<td>0.02</td>
<td>+ 1.5 \cdot 1.8</td>
<td>- 1.0</td>
<td>1.72</td>
</tr>
<tr>
<td>10yr</td>
<td>0.28</td>
<td>+ 1.5 \cdot 1.8</td>
<td>- 1.0</td>
<td>1.98</td>
</tr>
<tr>
<td>20yr</td>
<td>0.61</td>
<td>+ 1.5 \cdot 1.8</td>
<td>- 1.0</td>
<td>2.31</td>
</tr>
<tr>
<td>30yr</td>
<td>0.81</td>
<td>+ 1.5 \cdot 1.8</td>
<td>- 1.0</td>
<td>2.51</td>
</tr>
</tbody>
</table>
Here are estimates of the nominal social discount rate. In this, $\rho$ is nominal, and as explained above, $\theta$ is still the real rate of change of PCE per capita. Note that there is no 20-year Treasury bond, so we can’t directly estimate this case. In practice, you could interpolate between the 10-year and 30-year rates.

<table>
<thead>
<tr>
<th>Project Length</th>
<th>$\rho$</th>
<th>$+ \eta \cdot \theta$</th>
<th>$- \nu$</th>
<th>$= r^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5yr</td>
<td>1.33</td>
<td>$+ 1.5 \cdot 1.8$</td>
<td>$- 1.0$</td>
<td>$= 3.03$</td>
</tr>
<tr>
<td>10yr</td>
<td>2.10</td>
<td>$+ 1.5 \cdot 1.8$</td>
<td>$- 1.0$</td>
<td>$= 3.80$</td>
</tr>
<tr>
<td>20yr</td>
<td>NA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30yr</td>
<td>2.88</td>
<td>$+ 1.5 \cdot 1.8$</td>
<td>$- 1.0$</td>
<td>$= 4.58$</td>
</tr>
</tbody>
</table>
Next, let’s estimate the social opportunity cost of capital.

- Our starting point is the projected yield on US corporate AAA bonds, which are the safest corporate bonds.
- Public-sector projects do not pay taxes, but corporate bonds are paid out after corporate taxes have been paid.
- We therefore need to reverse this: we want the “pure”, ie pre-tax yield.
- From the US Statistical Abstract, the average corporate tax rate is about 23% (taxes paid as a percentage of corporate profits). We assume that this hold for the future as well. So we adjust (divide) the AAA return by $1 - 0.23 = 0.77$. 
Here are estimates of the social opportunity cost of capital. Note: there are no 30-year AAA corporate bonds, so we can’t estimate for this case.

<table>
<thead>
<tr>
<th>Project Length:</th>
<th>5yr</th>
<th>10yr</th>
<th>20yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal AAA rate</td>
<td>0.87%</td>
<td>2.75%</td>
<td>3.33%</td>
</tr>
<tr>
<td>Inflation</td>
<td>2.20%</td>
<td>2.00%</td>
<td>2.00%</td>
</tr>
<tr>
<td>Real AAA rate</td>
<td>1.33%</td>
<td>0.97%</td>
<td>1.33%</td>
</tr>
<tr>
<td>1— corp tax rate</td>
<td>0.77</td>
<td>0.77</td>
<td>0.77</td>
</tr>
<tr>
<td>Real Social Opp Cost</td>
<td>−1.72%</td>
<td>0.97%</td>
<td>1.73%</td>
</tr>
<tr>
<td>Nominal Social Opp Cost</td>
<td>1.33%</td>
<td>3.57%</td>
<td>4.33%</td>
</tr>
</tbody>
</table>
The table below summarizes our results, when we are working with real (constant purchasing-power) units.

<table>
<thead>
<tr>
<th>Project Length:</th>
<th>5yr</th>
<th>10yr</th>
<th>20yr</th>
<th>30yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>OMB</td>
<td>0.0%</td>
<td>1.0%</td>
<td>1.6%</td>
<td>1.9%</td>
</tr>
<tr>
<td>Individual rate</td>
<td>0.0%</td>
<td>0.3%</td>
<td>0.8%</td>
<td>0.8%</td>
</tr>
<tr>
<td>Social Rate</td>
<td>1.7%</td>
<td>2.0%</td>
<td>2.3%</td>
<td>2.5%</td>
</tr>
<tr>
<td>Social Opportunity Cost of Capital</td>
<td>−1.7%</td>
<td>1.0%</td>
<td>1.7%</td>
<td>NA</td>
</tr>
</tbody>
</table>
The table below summarizes our results when the analysis is done in nominal units.

<table>
<thead>
<tr>
<th>Project Length:</th>
<th>5yr</th>
<th>10yr</th>
<th>20yr</th>
<th>30yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>OMB</td>
<td>1.9%</td>
<td>3.0%</td>
<td>3.6%</td>
<td>3.9%</td>
</tr>
<tr>
<td>Individual rate</td>
<td>1.3%</td>
<td>2.1%</td>
<td>NA</td>
<td>2.9%</td>
</tr>
<tr>
<td>Social Rate</td>
<td>3.0%</td>
<td>3.8%</td>
<td>NA</td>
<td>4.6%</td>
</tr>
<tr>
<td>Social Opportunity Cost of Capital</td>
<td>1.1%</td>
<td>3.6%</td>
<td>4.3%</td>
<td>NA</td>
</tr>
</tbody>
</table>
Some Comparisons

- There is some variation in these rates
- I incline towards the social rate as providing the most reasonable practical answer.
- For one thing, it is closer to the individualistic rates.
- For another, the social opportunity cost approach seems to say that the government’s use of capital always displaces private capital utilization. Recent experience with capital markets suggests that this is debatable.
Let’s begin with the off-beat cases

**Zero discount rate**

- First, note that this is zero to 1 decimal place: even Huntington bank currently pays about 1/10-th of 1% on its checking account (i.e., 0.001%).
- But a zero discount rate is possible psychologically. When $r = 0$, we have $1 + r = 1$, so the slope of the budget constraint, and hence, in equilibrium, the slope of the intertemporal indifference curve, is $-1$.
- This means that in individual regards a dollar tomorrow as indifferent to a dollar today: there is no psychological time preference.
- For project analysis, it means that you *can* actually add quantities over time.
Negative discount rate

- First, note that this is for the social opportunity cost of capital, which is based on the assumption of a fixed investment pool. If you regard this assumption as dubious (you believe that government projects do not displace private-sector projects dollar-for-dollar), you may want to disregard it.

- But analysts do believe that time preference rates can be negative.

- A negative rate means that you are willing to pay an institution (a bank or, for government securities, the government) to hold your money for you.
For some foreign cases this might be reasonable: suppose you feel unsafe, so that you view keeping your assets in your house (where it will earn no return, but on the other hand won’t cost you anything either) as a bad alternative. Then you might be willing to pay someone safe to hold on to your money. In this case, your time preferences are characterized by negative discount rates.

But this seems implausible to me, at least large-scale behavior in the US case (though it might be plausible for crime-ridden communities).
As to the other cases:

- Choice of real rates are reasonably narrow: between about 0% and 2.5%. Many projects will be evaluated the same qualitative way (ie as to yes or no) by rates in this range.

- Looking at the OMB rate specifically, note that it is consistently below the social rate. This suggests that OMB may be accepting too many projects (assuming that projects have costs up-front and benefits in the out-years).

- For nominal rates the range is a bit larger: from about 1% (5-year opportunity costs) to about 4.5%.

- But even this range may not make a lot of qualitative difference to project results.

- Note that the OMB rate is again below the social rate, and most of the time below the social opportunity cost of capital (except for very short-term projects).
One general approach to this variation is to do sensitivity testing. That is, use a range of rates to evaluate your projects. If the answer is qualitatively the same under a variety of rates, then we can be reasonably confident that our decision is right. The problem comes when some rates in the range result in one decision, and others result in a different decision. In that case, all you can really do is focus on the various rates, try to get more accurate estimates, and decide which underlying theory is most appropriate for your community.
Computational Appendices
Quick-and-Easy Real Rate (I)

- If you need a quick-and-easy way to get a real discount rate from a nominal rate and an inflation rate, then one way to do it, *if the rates aren't too large*, is to compute: real rate = nominal rate − inflation rate.

- Why this works: Let \( \hat{r} \) be the nominal rate, \( r \) be the real rate and \( i \) be the inflation rate. Then we have:

\[
1 + \hat{r} = (1 + r)(1 + i)
\]

\[
= 1 + r + i + ri
\]

- Now suppose that both \( r \) and \( i \) are small (say, no greater than .07 each). Then the product is less than 0.0049. Since this is less than half a percentage point, if you need an estimate good to one decimal place, it can be neglected.
Then, neglecting $ri$, we have:

\[
1 + \hat{r} = 1 + r + i \\
\hat{r} = r + i \\
r = \hat{r} - i
\]

as stated.

But remember that this relies on $r$ and $i$ being small, on the order of at most 0.07 each.

But if both $r$ and $i$ were both on the order of 0.10 (10%) then their product would be 0.01 so you’d be making a 1% error.
Here is how to compute an average annual rate of change.

- Consider a quantity $x$ that changes from an initial value $x_0$ to a final value $x_T$ in $T$ years.
- We define the *average annual rate of change* as the value $r$ such that

$$x_0(1 + r)^T = x_T$$

that is, the rate such that, if $x$ began at $x_0$ and changed at the constant rate $1 + r$ (i.e., ignoring any fluctuations in the rate over the period and hence at an “average” rate) it would become $x_T$ in $T$ years.
To compute the rate, begin with the definition:

\[ x_0 (1 + r)^T = x_T \]

and manipulate it to solve for \((1 + r)\) :

\[
(1 + r)^T = \frac{x_T}{x_0} \\
1 + r = \left( \frac{x_T}{x_0} \right)^{1/T}
\]

and when you know \(1 + r\), then \(r\) can be read off by inspection.
Example:

- For the US population:
  - Value in 2000 \( (x_0) = 281,421,906 \)
  - Value in 2012 \( (x_T) = 313,914,040 \)
  - Time lapse \( (T) = 12 \) years

So:

\[
\frac{x_T}{x_0} = \frac{313914040}{281421906} = 1.1155
\]

\[
1 + r = (1.1155)^{1/12}
\]

\[
= 1.0092
\]

implying that the population \( (x) \) is growing at an average rate of about 0.92% per annum.
Data Notes

Here is a quick guide to the sources of data used in the calculations.

- **Population**: population projections are done by the Census Bureau. (Labor force projections are provided by the Bureau of Labor Statistics).
- **Inflation**: I use OECD estimates of the projected GDP deflator. Other choices (CPI, PPI) are also possible.
- **Yields on Treasury Securities**: most financial reporting sites (eg Bloomberg, Yahoo) provide these.
- **Yields on corporate bonds**: financial reporting sites.
- **Personal Consumption expenditures**: the BLS projects real PCE to 2022. I assume that, after that the rate of change is the average for 1992-2012. For nominal PCE I just add the appropriate inflation rate.