Continuous Compounding and Discounting

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Most real-world project analysis is carried out as we’ve been doing it, with the present value of $\Delta Z_t$ in period $t$ computed as $\Delta Z_t / (1 + r)^t$, where $r$ is the appropriate discount rate.

But in some problems it is convenient to work with continuous time: for example, in problems where we want to select the optimal time to do something. That requires calculus — differentiation — and that in turn requires that time be a continuous quantity.

This note derives the basic equation for continuous compounding and discounting, i.e., when time is continuous.

In a practical (or in a problem/exam) setting you can use either the discrete or the continuous version — your choice.
Single individual.
Banking system, with bank rate $r$.
Time is denoted by $t$. 
Suppose you leave $\Delta Z_0$ on deposit with the bank for 1 year.

If the bank computes interest annually, then at the end of the year you’ll have $\Delta Z_0(1 + r)$. 
Now suppose the bank decides to compute your interest every half year. Then it’ll use an interest rate of $r/2$ and at the end of the half year you’ll have $\Delta Z'_0 = \Delta Z_0 (1 + r/2)$.

At the end of the full year your half-year earnings will have been compounded again and you’ll then have:

\[
\Delta Z_1 = \Delta Z'_0 (1 + \frac{r}{2})
= \left( \Delta Z_0 (1 + \frac{r}{2}) \right) (1 + \frac{r}{2})
= \Delta Z_0 (1 + \frac{r}{2})^2
\]
Compounding III

- If they decide to compute interest three times a year you’ll end up with \( \Delta Z_0 (1 + r/3)^3 \).
- If they decide to do it every month you’ll have \( \Delta Z_0 (1 + r/12)^{12} \).
- We can see where this is going: the bank compounds more and more often, but the effective rate used in each compounding gets smaller and smaller.
- In the limit, the bank is compounding “all the time”. This is continuous compounding.
Numerical Example I

Let’s see if we can gather some numerical evidence as to what happens as the number of compoundings gets larger and larger. Take $\Delta Z_0 = 1$ and $r = 0.05$. Then we have:

<table>
<thead>
<tr>
<th>Frequency</th>
<th>$N$</th>
<th>$(1 + \frac{r}{N})^N$</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>1</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
<td>1.05116</td>
<td>0.00116</td>
</tr>
<tr>
<td>Weekly</td>
<td>52</td>
<td>1.05125</td>
<td>0.00008</td>
</tr>
<tr>
<td>Daily</td>
<td>365</td>
<td>1.05127</td>
<td>0.00002</td>
</tr>
</tbody>
</table>

(The ‘Difference’ column shows the change in the previous column).
As we look at these results we see that the number of compoundings seems to make less and less difference to the final answer. This suggests that if we continued to subdivide our year (compounding every half-day, then every hour etc), we’d get essentially the same answer. Thus turns out to be true, and there is a mathematical result that makes this precise.
It can be proved that as the number of compoundings gets larger and larger, the result gets closer and closer to a particular (limiting) value: specifically:

$$\lim_{N \to \infty} \left( 1 + \frac{r}{N} \right)^N = e^r$$

where $e$ is the transcendental number, $e = 2.7182818...$
We conclude that with continuous compounding, $1 will grow into $e^r$ after one year.

So in our previous example with $r = 0.05$ the result, as the number of compoundings becomes very large, is that our deposit of $1 will be worth $e^{0.05} = 1.0512711 after one year.

And with continuous compounding, an initial deposit of $\Delta Z_0$ will grow into $\Delta Z_0 e^r$ after one year.
If we left the money on deposit for 2 years then, with continuous compounding, we’d receive another year’s worth of interest on the compounded amount, so we’d have \((\Delta Z_0 e^r)e^r = \Delta Z_0 e^{2r}\).

And by the same logic, after \(t\) years we’d end up with \(\Delta Z_0 e^{rt}\).

So we see that after \(t\) years with continuous compounding at rate \(r\), \(\Delta Z_0\) will grow into 
\[\Delta Z_0 e^{rt}\]
Continuous Present Value

- We can use this to compute the present value of a quantity $\Delta Z_t$ received in year $t$, by the following reasoning:
- If the present value of $\Delta Z_t$ is denoted $\Delta Z_0$ then it must be true (given the banking system) that we should be indifferent between (1) getting $\Delta Z_0$ now and leaving it on deposit for $t$ years and (2) getting $\Delta Z_t$ in year $t$.
- That is, the present value must satisfy
  \[
  \Delta Z_0 e^{rt} = \Delta Z_1
  \]
- Then, solving for $\Delta Z_0$, we get:
  \[
  \Delta Z_0 = \Delta Z_1 / e^{rt} = \Delta Z_1 e^{-rt}
  \]
  which is the present value of $\Delta Z_t$ when time is continuous, ie when compounding takes place continuously.
Numerical Comparison

Here’s a table comparing the present value of $1 to be received in year \( t \) using the two methods:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( t )</th>
<th>Continuous</th>
<th>Discrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>.01</td>
<td>1</td>
<td>0.990 049 83</td>
<td>0.990 099 01</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.904 837 42</td>
<td>0.905 286 95</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.740 818 22</td>
<td>0.741 922 92</td>
</tr>
<tr>
<td>0.05</td>
<td>1</td>
<td>0.951 229 42</td>
<td>0.952 380 95</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.606 530 66</td>
<td>0.613 913 25</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.223 130 16</td>
<td>0.231 377 45</td>
</tr>
<tr>
<td>0.10</td>
<td>1</td>
<td>0.904 837 42</td>
<td>0.909 090 91</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.367 879 44</td>
<td>0.385 543 29</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.04 978 707</td>
<td>0.05 730 855</td>
</tr>
</tbody>
</table>

As we can see, the discrete case gives slightly larger answers, though they’re never far apart.
With an annuity we want to sum the successive present values, each of which has the same payment value.

With continuous time, the analog of summation is integration, so we have, for an annuity starting in period $t_1$ and lasting until period $t_2$, and where the annuity amount is $\Delta Z$ in each period:

\[
PV = \Delta Z \int_{t_1}^{t_2} e^{-rt} dt
\]

\[
= \frac{-\Delta Z}{r} \left( e^{-rt_2} - e^{-rt_1} \right)
\]

\[
= \Delta Z \left( \frac{e^{-rt_1} - e^{-rt_2}}{r} \right)
\]
For the special case where the annuity starts today, ie when $t_1 = 0$, this becomes:

\[ PV = \Delta Z \left( e^{-rt_1} - e^{-rt_2} \right) \frac{1 - e^{-rt_2}}{r} \]

And for the special case where the annuity begins today and lasts forever, we have:

\[ PV = \frac{\Delta Z}{r} \]
Annuities III

Consider an annuity of $1 per year, beginning in year 0, and lasting until year $T$. Our two methods of computing give, for the annuity’s present value:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$T$</th>
<th>Continuous</th>
<th>Discrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>10</td>
<td>9.5162582</td>
<td>10.471305</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>18.126925</td>
<td>19.045553</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>25.918178</td>
<td>26.807708</td>
</tr>
<tr>
<td>0.05</td>
<td>10</td>
<td>7.8693868</td>
<td>8.7217349</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>12.642411</td>
<td>13.46221</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>15.537397</td>
<td>16.372451</td>
</tr>
<tr>
<td>0.10</td>
<td>10</td>
<td>6.3212056</td>
<td>7.1445671</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.6466472</td>
<td>9.5135637</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>9.5021293</td>
<td>10.426914</td>
</tr>
</tbody>
</table>
A quantity $x$ grows from an initial value $x_0$ to a final value $x_1$ in $T$ years.

For the continuous case, the average annual rate of change is the $r$ that solves:

$$x_0 \ e^{rT} = x_1$$

Then:

$$e^{rT} = \frac{x_1}{x_0}$$

$$rT = \ln \left( \frac{x_1}{x_0} \right)$$

$$r = \frac{1}{T} \ln \left( \frac{x_1}{x_0} \right)$$
For the example of US population growth in the handout on the social discount rate, we had:

- Value in 1980 \( (x_0) = 106,940 \)
- Value in 2008 \( (x_1) = 154,287 \)
- Time lapse \( (T) = 28 \) years

So, in continuous time:

\[
r = \frac{1}{28} \ln \frac{154287}{106287} = \frac{1}{28} \ln 1.4516
\]

\[
= \frac{1}{28} 0.37266640
\]

\[
= 0.01331
\]

so \( x \) is growing at about 1.3% per annum.
## Formula Summary

<table>
<thead>
<tr>
<th>Concept</th>
<th>Continuous</th>
<th>Discrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV of $\Delta Z_t$ in year $t$</td>
<td>$\Delta Z_t e^{-rt}$</td>
<td>$\frac{\Delta Z_t}{(1+r)^t}$</td>
</tr>
<tr>
<td>PV of annuity, amount $\Delta Z$, from $t$ to $T$</td>
<td>$\Delta Z \left( \frac{1-e^{-rT}}{r} \right)$</td>
<td>$\Delta Z \left( \frac{s^T-s^{T+1}}{1-s} \right)$</td>
</tr>
<tr>
<td>Avg Ann change rate, $x_0$ to $x_1$ in $T$ years</td>
<td>$\frac{1}{T} \ln \left( \frac{x_1}{x_0} \right)$</td>
<td>$\left( \frac{x_1}{x_0} \right)^{1/T} - 1$</td>
</tr>
</tbody>
</table>

where, in the discrete annuity formula, $s = 1/1 + r$. 