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*Warning: I wrote this very quickly, and it almost certainly contains typos, if not outright errors. Please let me know of any that you find: I will post corrections to the course website.
1 Introduction

Many problems of interest to economists and planners can be phrased as optimization problems. Examples include:
A firm chooses a level of output to maximize profits

An individual chooses a consumption bundle to maximize utility, subject to a budget constraint

A public agency chooses its inputs in order to minimize the cost of producing a given level of output.

A government chooses taxation levels on goods in order to allow individuals to maximize utility, subject to taxes yielding a given amount of revenue.

A firm at a given location chooses its sales (market) area to maximize its profits.

We will be studying all these problems throughout the course. The purpose of these notes is to summarize the tools from the theory of unconstrained optimization and optimization subject to equality constraints that will allow us to do so. I will be assuming that most of this is a review for you: exceptions may be the notions of indirect objective functions and envelope theorems. I shall also briefly consider the conditions for optimization subject to inequality constraints, but this is just for completeness: we will not actually be using studying inequality-constrained problems.

2 Ground Rules

Our basic goal will be to develop an understanding of the behavior of the decision-makers we will be studying. To this end, we will generally assume that the relevant optimization problems are well-behaved in the following senses:

- All functions will be assumed to be continuous with (at least) continuous second (partial) derivatives.
- All functions will be assumed to have the necessary shapes as to guarantee a unique interior solution to the various optimization problems.
- When we discuss the solutions to economic problems, we will generally assume that those solutions are economically meaningful: they will be both real (ie not involving the imaginary unit $\sqrt{-1}$) and positive.
One reason for this is that we are interested primarily in how decision-makers change their behavior around optima when features of the environment change (this will be the notion of comparative statics, to be developed later). That is, we take for granted that the decision-maker has already found a meaningful optimal solution to his/her problem, and we proceed to study the impact of small changes in the environment on those solutions. In this sense, we will be looking at local changes in behavior.

3 Notation

- A function of \( n \) variables, \( x_1, x_2, \ldots, x_n \) will be written as \( f(x_1, x_2, \ldots, x_n) \) or more compactly as \( f(x) \) with the context indicating whether this is a vector-valued function or a function of a single variable. In textbooks the usual convention is to denote vectors by bold-faced letters, so \( f(x) \) would represent a function of one variable, while \( f(x) \) would be vector-valued. We will also be a bit sloppy about worrying about row- versus column-vectors. The usual convention is to consider all vectors to be column vectors, in which case certain vector operations need to be transposed; but we won’t worry about that here.

- We will usually want to distinguish between the independent variables of a problem, and the values of parameters, which are features of the problem that are to be considered as fixed by the decision-maker. For example, one might have a function like:

\[
x_1 p_1 + x_2 p_2 = M
\]

which you should recognize as an individual’s budget constraint in a 2-good world. The individual will choose consumption levels of the two goods \( (x_1 \) and \( x_2) \), the independent variables, subject to, as fixed parameters, the levels of prices \( (p_1 \) and \( p_2) \) and available income \( M \). We will write any function like this as:

\[
g(x; \alpha)
\]

where, here \( x = x_1, x_2 \) are the independent variables and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) = (p_1, p_2, M) \) are the parameters.

- Among the notations we will use are for the first partial derivatives of a function \( f(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_m) \) are:

\[
\frac{\partial f}{\partial x_i} \text{ and } f_i
\]

4
while the second partial derivatives are, eg,
\[ \frac{\partial^2 f}{\partial x_i \partial x_j} = f_{ij} \]

Remember that for \( f \) continuous we have, by Young's Theorem:
\[ f_{ij} = f_{ji} \]

ie the values of the mixed partials do not depend on the order in which you compute them.

4 Implicit Function Theorem

4.1 The problem and its solution

Suppose we are interested in the set of points \((x_1, x_2)\) that satisfy a relation like:
\[ g(x_1, x_2) = 0 \]

for example, specifically:
\[ 4x_1 + 3x_2 - 7 = 0 \]

In the specific case, we know how to proceed: we just solve the equation for (say) \( x_2 \) : this will be some function of \( x_1 \), say \( x_2^*(x_1) \), and in this specific case an easy calculation gives \( x_2^*(x_1) = (7 - 4x_1)/3 \). Note that this function will satisfy our equation exactly, for any value of \( x_1 \) (it is an identity in \( x_1 \)) and if we plug in \( x_2^* \) into the equation we have:
\[ 4x_1 + 3 \left( \frac{7 - 4x_1}{3} \right) - 7 = 0 \]

which is obviously zero. We can also easily compute how \( x_2 \) must change as \( x_1 \) does:
\[ \frac{dx_2^*}{dx_1} = -\frac{4}{3} \]

So far, so good. But what if \( g(x_1, x_2) \) was a bit more complicated? For example, what if we were interested in
\[ \ln(x_1 + x_2) + x_1 x_2 - 12 = 0 \quad (x_1, x_2 > 0) \quad ? \]
Then we’d be in trouble since it’s not going to be possible to solve this for some explicit function \( x_2^*(x_1) \). And if we want to know what \( dx_2^*/dx_1 \) is, it looks like we’re out of luck.

The Implicit Function Theorem is an approach to this problem. It provides a set of conditions under which there exists a function like \( x_2^*(x_1) \) — even if we may not be able to display it explicitly — and provides a simple formula for computing \( dx_2^*/dx_1 \).

The statement of the Theorem for the two-variable case is as follows. Suppose \( g(x_1, x_2) \) is continuous and continuously differentiable around some point \((x_1^0, x_2^0)\) that satisfies \( g(x_1^0, x_2^0) = 0 \). Then there is a region around this point and a function \( x_2^*(x_1) \) such that in that region \( g(x_1, x_2^*(x_1)) = 0 \). Moreover, if \( g_2 \equiv \partial g/\partial x_2 \neq 0 \), we have:

\[
\frac{dx_2^*}{dx_1} = -\frac{g_1}{g_2} \quad \left( = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2} \right)
\]

We will regularly appeal to this theorem in our work. In terms of our ground rules, since we’ve already assumed that all functions are continuous and differentiable, the conditions of the Implicit Function Theorem amount to adding that \( g_2 \neq 0 \).

The Theorem also generalizes to more than two variables. We will have \( n \) equations in \( n \) unknowns and a set \( \alpha \) of parameters. The theorem asserts that we can solve the equations as functions of the parameters. The condition for being able to do this (the analog of the condition \( \partial g/\partial x_2 \neq 0 \) in the simple case) is that the Jacobian matrix of the equations (the matrix whose \( i, j \)-entry is the first partial derivative if the \( i \)-th equation with respect to \( x_j \)) is non-singular. We will be assuming this, too.

### 4.2 Application : slope of a level curve

Suppose we have a function \( g(x_1, x_2) \) and we want to examine its behavior for points \( x_1 \) and \( x_2 \) that satisfy (say) \( g(x_1, x_2) = k \) where \( k \) is a constant. (Note that we can make this fit our previous setup by working with \( h(x_1, x_2) = 0 \), where \( h(x_1, x_2) = g(x_1, x_2) - k \), but since \( k \) is a constant, nothing will depend on the way we write our problem). Points that satisfy this equation are said to be on a level curve (level surface in higher dimensions) of the function. (Think of generating points on a given indifference curve). In particular, we want to know how \( x_1 \) must
change when \( x_2 \) changes, if we are to remain on \( g(x_1, x_2) = k \). That is, we are looking for the rate of change of \( x_1 \) with respect to \( x_2 \) along \( g(x_1, x_2) = k \), which you sometimes see written as:

\[
\frac{dx_1}{dx_2} \bigg|_{g(x_1, x_2) = k}.
\]

Note that the answer is not given by a partial derivative of \( g \) because we have for example:

\[
\frac{\partial g}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{g(x_1 + \Delta x_1, x_2) - g(x_1, x_2)}{\Delta x_1}
\]

and there’s nothing in this that forces us to stay on \( g(x_1, x_2) = k \), which is what we want to do.

You may think that at this level of generality there’s not much we can say, but this is wrong. Assuming that \( g \) satisfies the conditions of the Implicit Function Theorem, there is a function, call it \( x_1^*(x_2, k) \) such that:

\[
g(x_1^*(x_2, k), x_2) = k \quad \text{for any } x_1
\]

(in other words, we can in principle solve \( g(x_1, x_2) = k \) for \( x_1 \), and the solution will depend on \( x_2 \) and \( k \)). Since this is an identity in \( x_2 \) we can differentiate both sides with respect to \( x_2 \), and we find:

\[
\frac{\partial g}{\partial x_1} \frac{dx_1^*}{dx_2} + \frac{\partial g}{\partial x_2} = 0
\]

so, assuming that \( \partial g / \partial x_1 \neq 0 \) and re-arranging:

\[
\frac{dx_1^*}{dx_2} = -\frac{\partial g / \partial x_2}{\partial g / \partial x_1}
\]

or, in a slightly more compact notation:

\[
\frac{dx_1^*}{dx_2} = -\frac{g_2}{g_1}
\]

By the same token, looking at it the other way round, and solving instead for \( x_2 \), we have:

\[
\frac{dx_2^*}{dx_1} = -\frac{g_1}{g_2}
\]

assuming that \( \partial g / \partial x_2 \neq 0 \).

This is a pattern worth bearing in mind: when you see (minus) the ratio of two partial derivatives you should immediately think: slope of a level curve.
5 Matrix Algebra

5.1 Linear equation systems

We assume that the matrix notation for systems of linear equations is familiar to you. If $A$ is an $n \times n$ matrix, and $x$ and $b$ are $n \times 1$ vectors, then the matrix equation:

$$Ax = b$$

represents a linear system of $n$ equations in the unknowns $x$. If $A$ is non-singular (meaning: the inverse $A^{-1}$ exists, so that the determinant of $A$, det($A$) which we will generally write as $|A|$, is nonzero), the solution is given by:

$$x = A^{-1}b$$

obtained by pre-multiplying the equation on both sides by $A^{-1}$. (Remember that matrix multiplication is non-commutative: multiplying by a matrix on one side is generally not the same as multiplying on the other: $AB \neq BA$).

5.2 Cramer’s rule

We will also need an alternative way to develop the solutions $x$ to the matrix equation. Let $A_{i,b}$ be the result of replacing column $i$ of $A$ with the elements of $b$. Then Cramer’s Rule says that the $i$-th element of the solution vector, $x_i$, is the ratio of two determinants:

$$x_i = \left| A_{i,b} \right| / |A|$$

Example: let

$$A = \begin{bmatrix} 4 & 7 & 12 \\ 3 & 8 & 6 \\ 2 & 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

A simple computation shows that det($A$) = $-52$, so $A$ is invertible and it turns out that:

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{4}{11} & \frac{22}{52} \\ 0 & \frac{1}{11} & \frac{-2}{52} \\ \frac{1}{4} & \frac{-5}{26} & \frac{11}{52} \end{bmatrix}$$
and then we have:

\[
x = A^{-1}b
\]

\[
= \begin{bmatrix}
-\frac{1}{2} & \frac{13}{2} & \frac{27}{26} \\
0 & \frac{13}{12} & \frac{11}{13} \\
\frac{1}{4} & -\frac{26}{13} & \frac{13}{13}
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
= \begin{bmatrix}
\frac{42}{13} \\
\frac{-52}{13} \\
\frac{5}{13}
\end{bmatrix}
\]

We now do the same thing using Cramer’s rule. Focussing on the element \(x_2\), the rule tells us to form

\[
A_{2,b} = \begin{bmatrix}
4 & 1 & 12 \\
3 & 2 & 6 \\
2 & 3 & 4
\end{bmatrix}
\]

by inserting \(b\) into column 2 of \(A\); and then \(x_2\) is given by:

\[
x_2 = \frac{|A_{2,b}|}{|A|}
\]

\[
= \frac{20}{-52}
\]

\[
= \frac{5}{13}
\]

as before.

### 5.3 Minors

Let \(H\) be a square \(n \times n\) matrix. A minor of \(A\) is the determinant of the matrix that remains when we remove any row and any column from \(A\). However, we will not use minors directly, but instead focus on an extension of this idea.

A principal minor of order \(k\) of \(A\) is the determinant of the matrix that remains when we remove any \(n - k\) rows and the corresponding numbered columns.

So for example, if we decide to remove rows 1 and 4, then we obtain a principal minor if we also remove columns 1 and 4 (and take the determinant of what remains).

A principal minor of order \(k\) has us remove \(n - k\) rows and columns. Its size is therefore \(n - (n - k) = k\). So a principal minor of order \(k\) is the determinant of a \(k \times k\) matrix.
One more idea: a leading principal minor of order $k$ (sometimes called a naturally-ordered principal minor of order $k$) is the determinant of the matrix that remains when we remove the last $n-k$ rows and the corresponding columns. Since there is only one way to do this, there is just one leading principal minor of order $k$ for each matrix.

Example: suppose that $A$ is a $4 \times 4$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

We can create a principal minor of order $k = 2$ by removing any $n-k = 4-2 = 2$ rows and the corresponding columns. If we pick rows and columns 2 and 3 we get:

$$A_2 = \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}$$

But we could also pick rows and columns 1 and 4, so:

$$A_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is another principal minor of order 2. But there is only one way to remove the last $k = 2$ rows and columns, so that:

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is the only leading principal minor of order 2. For our example, you should be able to convince yourself that all the leading principal minors are:

$$A_1 = |a_{11}| \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad A_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

That is, you can form the leading principal minors by starting with the upper left-hand ($= (1, 1)$) element and gradually adding more rows and columns immediately to the right of the selected ones.

Alternatively, start with the whole matrix, and make it smaller by successively removing more rows and columns, starting from the left. (In terms of the display, start with $A_4$ and work backwards).
5.4 The Hessian

The Hessian matrix of a function of \( n \) independent variables \( f(x_1, x_2, \ldots, x_n; \alpha) \) is the matrix of second derivatives of \( f \) with respect to the independent variables. That is:

\[
H_f = \begin{bmatrix}
  f_{11} & f_{12} & \cdots & f_{1n} \\
  f_{21} & f_{22} & \cdots & f_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix}
\]

We will usually omit naming the function in question when there’s no room for doubt, so in the above case we’d just use \( H \) (instead of \( H_f \)) to refer to the Hessian of the function \( f \).

6 Unconstrained Optimization

We turn finally to optimization proper.

Let \( f(x; \alpha) = f(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_m) \) be a function of \( n \) independent variables (the \( x \)'s) and \( m \) parameters (the \( \alpha \)'s). The canonical problem of unconstrained optimization is to choose the \( x \)'s to optimize (ie, maximize or minimize) \( f \). In this context, the \( x \)'s are often called the decision variables or the choice variables, and \( f \) is the optimand or objective function. We write the problem as

\[
\max/\min_x f(x; \alpha)
\]

6.1 First-order conditions

A necessary condition for a vector \( x^* = (x_1^*, x_2^*, \ldots, x_n^*) \) to solve the optimization problem is contained in the first-order conditions, or FOCs, namely that the first (partial) derivatives of \( f \) all vanish. We write this as:

\[
f_i \equiv \frac{\partial f}{\partial x_i} = 0 \quad \text{for } i = 1, 2, \ldots, n
\]

A point satisfying the FOCs is called a stationary point of \( f \). As you will remember, this is a necessary condition for an optimum: if \( x^* \) solves the problem, then at \( x^* \) the FOCs hold. However, the converse is not true. A point satisfying the FOCs
can be a maximum, a minimum or neither (called a saddle point) as illustrated in the following picture for the case where \( n = 1 \):

Here the FOC holds at \( x_1 \) (a minimum), \( x_2 \) (a maximum) and at \( x_3 \) (neither a max nor a min). How can we tell which is which?

### 6.2 Second-order conditions

If we examine the picture, it should be geometrically clear how to distinguish a maximum from a minimum. At a minimum point like \( x_1 \) we see that the slope (first derivative) of \( f \) is negative to the left of \( x_1 \) and positive to the right. So when we have a minimum, the first derivative goes from negative to positive around the stationary point, i.e., the slope of the first derivative is increasing. Another way of saying this is that the second derivative (the slope of the first derivative) is increasing. At a maximum (like \( x_3 \)) the first derivative is positive to the left of the stationary point and negative to the right of it, i.e., the slope of first derivative is decreasing around \( x_3 \). In other words, the second derivative is decreasing. At a saddle point like \( x_2 \) the first derivative is positive (or more generally, has the same sign) both to the right and to the left of the point.

So for a function \( f(x) \) of a single independent variable, our test is: if \( x^* \) is a stationary point of \( f \) satisfying \( f'(x^*) = df/dx = 0 \) then:

- **Maximum:** If \( f''(x^*) = d^2 f/dx^2 < 0 \) then \( f \) has a maximum at \( x^* \).
- **Minimum:** If \( f''(x^*) = d^2 f/dx^2 > 0 \) then \( f \) has a minimum at \( x^* \).
For functions of many variables, the corresponding condition involves the principal minors of the Hessian matrix. Specifically: at a stationary point \( x^* \) satisfying the FOCs:

- **Maximum:** If the principal minors of order \( k \) of the Hessian matrix of \( f \) have sign \((-1)^k\) then \( x^* \) is a maximum point. Another way of putting this is that the principal minors alternate in sign, beginning with the negative.

- **Minimum:** If the principal minors of order \( k \) are all positive for all \( k \), then \( x^* \) is a minimum point of \( f \).

- If neither of these hold (including the case where some of the principal minors are zero) then we have a saddle-point.

These are the *second-order conditions* (SOCs) for an optimum.

For our work on comparative statics, we will be looking to identify certain expressions as being principal minors of the Hessian matrix. So we could stop here. But for practical optimization, where we are interested in checking whether a (numerical) stationary point is a maximum, a minimum, or neither, there is a shortcut. Recall that there will usually be many principal minors of a given order. But it can be shown that if the *leading* principal minors — of which there is just one for each order — have the corresponding property, then all the principal minors do. So a labor-saving statement of the SOCs is:

- **Maximum:** If the leading principal minor of order \( k \) of the Hessian matrix of \( f \) has sign \((-1)^k\) then \( x^* \) is a maximum point. Another way of putting this is that the leading principal minors alternate in sign, beginning with the negative.

- **Minimum:** If the leading principal minor of order \( k \) is positive, for all \( k \), then \( x^* \) is a minimum point of \( f \).

- If neither of these hold (including the case where some of the principal minors are zero) then we have a saddle-point.
6.3 Example

Consider the problem of optimizing

\[ f(x_1, x_2) = x_1^2 + x_2^2 + 0.5x_1x_2 + 4x_2 \]

The FOCs are:

\[
\frac{\partial f}{\partial x_1} = 0 : 2x_1 + 0.5x_2 = 0 \\
\frac{\partial f}{\partial x_2} = 0 : 2x_2 + 0.5x_1 + 4 = 0
\]

From the first FOC we have:

\[
2x_1 = -0.5x_2 \\
x_1 = -\frac{x_2}{4}
\]

and then, inserting this into the second FOC we get:

\[
2x_2 - \frac{1}{2} \left( \frac{x_2}{4} \right) = -4 \\
2x_2 - \frac{x_2}{8} = -4
\]

or:

\[
x_2^* = \frac{-32}{15} = -2.1333
\]

and then:

\[
x_1^* = -\frac{32}{60} = \frac{8}{15} = 0.5333
\]

So \((x_1^* = 8/15, x_2^* = -32/15)\) is a stationary point of \(f\). What kind of stationary point is it? We look at the Hessian, which for this problem is simple: differentiating the FOCs variable by variable we find that:

\[
H = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2}
\end{bmatrix} = \begin{bmatrix}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{bmatrix} = \begin{bmatrix}
2 & 1/2 \\
1/2 & 2
\end{bmatrix}
\]

The leading principal minor of order 1 is the \(H_{11}\) term, which here is \(2 > 0\), while the determinant of the entire Hessian (which is the leading principal minor of order 2) is \(4 - (1/2)^2 = 15/4 > 0\). Thus all the leading principal minors are positive, so we have found that \((x_1^* = 8/15, x_2^* = -32/15)\) is a minimum point of \(f\).
6.4 The indirect objective function

Consider again our basic problem:

$$\max/\min_x f(x; \alpha)$$

Assuming that the conditions for the Implicit Function Theorem hold, we can solve the FOCs for the $x$’s, and the solutions will depend on the problem parameters. We write them as:

$$x_i^*(\alpha) \quad i = 1, 2, \ldots, n$$

Note that by the conditions of the Implicit Function Theorem, the solution functions will satisfy the FOCs as identities: that is, they will hold for any values of the parameters.

How well have we done with respect of our objective function? We just evaluate the objective function at the optimal choices for the $x$’s, and obtain:

$$f(x_1^*(\alpha), x_2^*(\alpha), \ldots, x_n^*(\alpha); \alpha) \equiv f^*(\alpha)$$

called the indirect objective function for our problem. Note that the indirect objective function depends only on the problem parameters (the $\alpha$’s) because the $x^*$’s are functions, not variables. Thus we have:

$$\frac{\partial f(x_1^*(\alpha), x_2^*(\alpha); \alpha)}{\partial x_i} = 0 \quad \forall \alpha, \quad i = 1, 2, \ldots, n$$

For example, consider a slight generalization of the problem we’ve just solved: optimize

$$f(x_1, x_2) = x_1^2 + x_2^2 + 0.5x_1x_2 + bx_2$$

where now parametrize the problem in terms of $b$ (via the term $bx_2$) at the end of the optimand. The FOCs are:

$$f_1 = 0 : 2x_1 + 0.5x_2 = 0$$
$$f_2 = 0 : 2x_2 + 0.5x_1 + b = 0$$

As before we have $x_1 = -x_2/4$ by solving the first FOC, and then, inserting this
into the second FOC we have:
\[
2x_2 - \frac{x_2}{8} = -b \\
15 \cdot \frac{8}{x_2} = -b \\
x_2 = -b \cdot \frac{8}{15} = -0.533 33b = x_2^*(b) \\
x_1 = -\left(-b \cdot \frac{8}{60}\right) = \frac{1}{15} b = 0.133 33b = x_1^*(b)
\]
The point is that both solutions depend on the problem parameters, here just \(b\). The indirect objective function is:

\[
f^*(b) = \left[x_1^*(b)\right]^2 + \left[x_2^*(b)\right]^2 + 0.5x_1^*(b)x_2^*(b) + b x_2 \\
= \left(\frac{2}{15} b\right)^2 + \left(b \cdot \frac{8}{15}\right)^2 + \frac{1}{2} \left(\frac{2}{15} b\right) \left(-b \cdot \frac{8}{15}\right) + b \left(-b \cdot \frac{8}{15}\right) \\
= -\frac{4}{15} b^2
\]
after a bit of routine simplification. The point again is that the value of the indirect objective function depends only on the problem parameters.

### 6.5 The Envelope Theorem for unconstrained optimization

A natural question to ask at this point is: how does our best answer (the value of the indirect objective function) change as the values of the parameters change? That is, we are asking about the value of:

\[
\frac{\partial f^*(a)}{\partial a_j} = \frac{\partial}{\partial a_j} f(x_1^*(a), x_2^*(a), \ldots, x_n^*(a); a)
\]
at the optimizing point \(x^*\). This is routine computation, too, using the chain rule of differentiation, and we find:

\[
\frac{\partial f^*(a)}{\partial a_j} = \frac{\partial}{\partial a_j} f(x_1^*(a), x_2^*(a), \ldots, x_n^*(a); a) \\
= f_1 \frac{\partial x_1^*}{\partial a_j} + f_2 \frac{\partial x_2^*}{\partial a_j} + \cdots + f_n \frac{\partial x_n^*}{\partial a_j} + \frac{\partial f}{\partial a_j}
\]

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(the last term arises because $f$ depends on $a_j$ directly as well as through the optimal solution functions). But at the stationary point $x^*(\alpha)$ we have $f_i = 0$ by the FOCs, so all terms like $f_n \partial x_n^* / \partial \alpha_j$ are zero and:

$$\frac{\partial f^*(\alpha)}{\partial \alpha_j} = \frac{\partial f}{\partial \alpha_j} \text{ at } x^*(\alpha)$$

That is, the rate of change of the indirect objective function with respect to a problem parameter near the optimizing point, is given by the rate of change (partial derivative) of the original optimand (evaluated at $x^*$).

What this means is that, if all you want to know is how the indirect objective function changes as the parameters change, you do not have to compute the indirect objective function first. All you need to do is differentiate the (original) optimand with respect to the parameter in question, and evaluate all answers at the optimal solutions.

Continuing the example of the problem: optimize

$$f(x_1, x_2) = x_1^2 + x_2^2 + 0.5x_1x_2 + bx_2$$

and suppose we want to know how the value of the indirect objective function changes as the value of $b$ changes. We could go through the whole rigmarole of solving the problem, obtaining the solutions, plugging them into the objective function, and then differentiating. But the Envelope Theorem tells us that all we need to do is differentiate the original objective function, and evaluate the result at the optimal point. That is, we see that:

$$\frac{\partial f^*(b)}{\partial b} = \frac{\partial f}{\partial b} = \frac{\partial}{\partial b} (x_1^2 + x_2^2 + 0.5x_1x_2 + bx_2) = x_2^*$$

almost by inspection. In this case, of course, we have done the computation the long way and we found

$$f^*(b) = -\frac{4}{15}b^2$$

so, directly:

$$\frac{\partial f^*}{\partial b} = -\frac{8}{15}b = x_2^*(b)$$

as we’ve seen.

You may say: using the Envelope Theorem isn’t all that much help if we really want to know how $f^*$ changes as $b$ changes, since we still need to know something
about $x_2^*$. In a sense, this is true. But many times what we will be interested in is the direction (sign) of the change in $f^*$ as a parameter changes: does increasing [decreasing] the value of a parameter increase or decrease the value of the indirect objective function? And in many economics/planning problems we will be able to assume that we know the signs of the solutions (in the real world, they will be positive: it’s going to be difficult to consume a negative amount of ice-cream, for example). In this case we may be able to use the Envelope Theorem to establish the direction (sign) of the indirect objective function by these considerations alone.

In our example, if $b > 0$ (corresponding to the numeric case we looked at first, where $b$ was 4) then $x_2^* < 0$ and we therefore know that:

$$\frac{\partial f^*}{\partial b} < 0 \quad \text{(given that } b > 0)$$

(with the direction changing if $b$ had the opposite sign). We don’t get complete detailed information, but as we will see later in the course, the information delivered by the Envelope Theorem will allow us to get a useful understanding of the behavior of solutions to many optimization problems.

7 Equality-Constrained Optimization

We turn now to our second class of optimization problem. We consider the problem:

$$\text{max} / \text{min}_{x} \quad f(x; a)$$

subject to $g(x; a) = b$

where the crucial feature of the constraint is that we require it to hold as an equality. Note that here we have just one constraint. We will see later how this will generalize.

7.1 Solution via Lagrange multipliers

One standard way of attempting to solve the problem is via the method of Lagrange Multipliers, which consists of the following steps:

- Write down a new function called the Lagrangian, based on the original op-
timand and the constraint, as follows:

\[ \mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x; \alpha)) \]

where we have introduced a new variable \( \lambda \) called a Lagrange multiplier. Note that we write the constraint as \( b - g(x; \alpha) \) and we add it to the original optimand \( f(x) \): see section 7.14 for a discussion of alternative ways of setting up the Lagrangian.

- Now optimize \( \mathcal{L}(x, \lambda; \alpha) \) as a *unconstrained* optimization problem in the original independent variables \( x (= x_1, x_2, \ldots, x_n) \) and the new variable \( \lambda \).

Thus, at the cost of a bit of complication induced by the new variable \( \lambda \), we have reduced the problem of equality-constrained optimization to one of unconstrained optimization, which we already know how to solve. We can refer to \((x^*, \lambda^*)\) as the *augmented* set of choice-variables. See Section 7.15 for the theoretical basis of the Lagrangian technique.

### 7.2 The FOCs

The FOCs (necessary conditions) for \((x^*, \lambda^*)\) to be a stationary point of:

\[ \mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x; \alpha)) \]

is that the first partial derivatives with respect to the augmented choice variables variables vanish. That is, we have:

\[
\frac{\partial \mathcal{L}(x, \lambda; \alpha)}{\partial x_i} = 0 : f_i - \lambda g_i = 0 \quad i = 1, 2, \ldots, n
\]

\[
\frac{\partial \mathcal{L}(x, \lambda; \alpha)}{\partial \lambda} = 0 : b - g(x; \alpha) = 0
\]

Note that the final condition just gives us back the original constraint: this tells us that in order to solve the original problem we must satisfy the constraint. To find a stationary point we appeal to the Implicit Function Theorem, and assert that we can solve the FOCs for the augmented choice variables: as before, these will depend on the problem parameters, and we can write them as:

\[
x_i^*(\alpha)
\]

\[
\lambda^*(\alpha)
\]

Note again that these will satisfy the FOCs as identities in \( \alpha \).
7.3 Example

Consider the problem:

\[
\begin{align*}
\max/\min_x & \quad f(x) = x^2 + y^2 \\
\text{s.t} & \quad x + y = 4
\end{align*}
\]

If you think about the problem for a moment you should be able to solve it (ie find a stationary point) in your head without all the mechanics of Lagrange multipliers, since \(x\) and \(y\) are treated symmetrically in both \(f\) and \(g\), and they should add up to 4. Nevertheless, let’s use it as an excuse to try out the Lagrange multiplier method. The Lagrangian is:

\[
\mathcal{L}(x, y, \lambda) = x^2 + y^2 + \lambda(4 - x - y)
\]

and we have the FOCs:

\[
\begin{align*}
\mathcal{L}_x(x, y, \lambda) &= 0 : 2x - \lambda = 0 \\
\mathcal{L}_y(x, y, \lambda) &= 0 : 2y - \lambda = 0 \\
\mathcal{L}_\lambda(x, y, \lambda) &= 0 : 4 - x - y = 0
\end{align*}
\]

From the first two we see that:

\[
2x = 2y
\]

so \(x = y\); and then, inserting this into the third FOC we get:

\[
\begin{align*}
4 &= 2y \\
y &= 2 \\
x &= 2 \\
\lambda &= 4
\end{align*}
\]

So \((x^* = 2, y^* = 2, \lambda^* = 4)\) is a stationary point of the Lagrangian.

7.4 The bordered Hessian

Since we have reduced the problem to equality-constrained optimization to one of unconstrained optimization via the Lagrangian, we know that a stationary point could be a maximum, a minimum or neither. How can we tell which it is? The answer turns out to be a bit more complicated than in the unconstrained case, because
of the need satisfy the constraints. In other words, when we look at slopes around a stationary point, we need to ensure that we do so while maintaining the equality constraint.

Begin with the FOCs of the Lagrangian $L(x, \lambda; \alpha)$ and differentiate them again with respect to the augmented choice variables (i.e., including $\lambda$). The result is:

\[
H = \begin{bmatrix}
L_{x_1 x_1} & L_{x_1 x_2} & \cdots & L_{x_1 x_n} & L_{x_1 \lambda} \\
L_{x_2 x_1} & L_{x_2 x_2} & \cdots & L_{x_2 x_n} & L_{x_2 \lambda} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
L_{x_n x_1} & L_{x_n x_2} & \cdots & L_{x_n x_n} & L_{x_n \lambda} \\
L_{\lambda x_1} & L_{\lambda x_2} & \cdots & L_{\lambda x_n} & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
f_{11} - \lambda g_{11} & f_{12} - \lambda g_{12} & \cdots & f_{1n} - \lambda g_{1n} & -g_1 \\
f_{21} - \lambda g_{21} & f_{22} - \lambda g_{22} & \cdots & f_{2n} - \lambda g_{2n} & -g_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n1} - \lambda g_{n1} & f_{n2} - \lambda g_{n2} & \cdots & f_{nn} - \lambda g_{nn} & -g_n \\
-g_1 & -g_2 & \cdots & -g_n & 0
\end{bmatrix}
\]

This is known as the bordered Hessian matrix, and we can see why: we start with the Hessian computed as the second partial derivatives of $L$ with respect to the independent variables (the $x$’s) and then we border this with (minus) the first partial derivatives of the constraint function $g$; and the $(n, n)$ element in the lower-right corner is zero.

Note that for our optimization problem with $n$ choice variables and 1 constraint, the bordered Hessian will be an $(n + 1) \times (n + 1)$ square matrix.

### 7.5 Border-preserving principal minors

When we considered the max/min question in the case of unconstrained optimization we looked at the principal minors of the Hessian. In the case of equality-constrained optimization it’s the same sort of thing, with a twist.

We define a border-preserving principal minor of order $k$ as the determinant of the bordered Hessian when you remove $n - k$ rows and the corresponding columns (as before), with the added proviso that you do not remove the entire border. Of course, when you remove rows and columns you will be removing elements of the
border; but you are not allowed to remove the entire border. In other words, in selecting rows and columns to remove, you cannot choose the last one.

If you remove \( n - k \) rows and columns, then given that the bordered Hessian is an \((n + 1)\) square matrix, what’s left will have \((n + 1) - (n - k) = k + 1\) rows and columns.

Thus, in a 1-equality-constraint optimization problem, a border-preserving principal minor of order \( k \) will be the determinant of a \((k + 1) \times (k + 1)\) square matrix.

### 7.6 SOCs: one equality constraint

We are now in a position to state the SOCs for a 1-equality-constraint problem. Let \((x^*, \lambda^*)\) be a stationary point of the Lagrangian, ie one satisfying the FOCs. Then

- **Maximum:** If the border-preserving principal minors of order \( k > 2 \) have sign \((-1)^k\) then \( f \) has a maximum at \( x^* \) subject to the constraint \( g = b \).

- **Minimum:** If the border-preserving principal minors of order \( k > 2 \) have sign \(-1\), ie are all negative, then \( f \) has a minimum at \( x^* \) subject to the constraint \( g = b \).

- Otherwise \( x^* \) is a saddle point.

As with the unconstrained case, there is a simplification. Define the *leading border-preserving principal minors of order \( k \)* as the determinant of the matrix formed when you remove the last \( n - k \) rows and columns, remembering that you cannot remove the entire border. As before, while there can be many border-preserving principal minors of order \( k \), there is just one *leading* border-preserving principal minor of order \( k \). Then we have:

- **Maximum:** If the leading border-preserving principal minor of order \( k > 2 \) has sign \((-1)^k\) then \( f \) has a maximum at \( x^* \) subject to the constraint \( g = b \).

- **Minimum:** If the leading border-preserving principal minor of order \( k > 2 \) has sign \(-1\) ie is negative, then \( f \) has a minimum at \( x^* \) subject to the constraint \( g = b \).

- Otherwise \( x^* \) is a saddle point.
7.7 Example revisited

In section 7.3 we considered the problem:
\[
\begin{align*}
\text{max/min} \quad & f(x) = x_1^2 + x_2^2 \\
\text{s.t} \quad & x_1 + x_2 = 4 
\end{align*}
\]
and found that \((x_1^*, x_2^* = 2, \lambda^* = 4)\) is a stationary point of the Lagrangian. What kind of stationary point is it? We compute the bordered Hessian, which in this case turns out to be:
\[
H = \begin{bmatrix}
2 & 0 & -1 \\
0 & 2 & -1 \\
-1 & -1 & 0 
\end{bmatrix}
\]
and with 1 equality constraint we want to check the leading border-preserving principal minor of order \(k > 2\). This will be a matrix of size \(3 \times 3\), ie in this case the entire bordered Hessian. It turns out that the determinant is \(-4\), so we conclude that this stationary point is a minimum.

7.8 Indirect objective function

As before, we are interested in knowing how well we have done. If \((x^*(a), \lambda^*(a))\) is a stationary point for the Lagrangian, then we define the indirect objective function, just as in the unconstrained case, as:
\[
f(x_1^*(a), x_2^*(a), \ldots, x_n^*(a); a) \equiv f^*(a)
\]

7.9 Envelope Theorem for equality-constrained optimization

We may want to understand how the indirect objective function changes as the value of a parameter, say \(a_j\), changes around an optimal solution. So, using the chain rule we differentiate the indirect objective function with respect to \(a_j\):
\[
\frac{\partial f^*}{\partial a_j} = f_1 \frac{\partial x_1^*}{\partial a_j} + f_2 \frac{\partial x_2^*}{\partial a_j} + \cdots + f_n \frac{\partial x_n^*}{\partial a_j} + \frac{\partial f}{\partial a_j}
\]
just as before. But now we face a problem. In the equality-constrained case, it is no longer true that the partial derivatives \(f_i\) are all zero as they were in the unconstrained case. So we can’t just cancel all of these.
But there is a trick. We know that our solution functions satisfy the constraints exactly, so:

$$g(x_1^*(\alpha), x_2^*(\alpha), \ldots, x_n^*(\alpha); \alpha) = 0$$

for any $\alpha$.

Therefore we can differentiate across both sides with respect to $\alpha_j$ and this will preserve the equality. If we do this we get:

$$g_1 \frac{\partial x_1^*}{\partial \alpha_j} + g_2 \frac{\partial x_2^*}{\partial \alpha_j} + \cdots + g_n \frac{\partial x_n^*}{\partial \alpha_j} + \frac{\partial g}{\partial \alpha_j} = 0$$

The equality will be preserved if we multiply it by some $\lambda$, so:

$$\lambda g_1 \frac{\partial x_1^*}{\partial \alpha_j} + \lambda g_2 \frac{\partial x_2^*}{\partial \alpha_j} + \cdots + \lambda g_n \frac{\partial x_n^*}{\partial \alpha_j} + \lambda \frac{\partial g}{\partial \alpha_j} = \lambda \cdot 0 = 0$$

But since this entire expression is zero, we can subtract it from the expression for $\frac{\partial f^*}{\partial \alpha_j}$ without changing anything. If we do this, we get:

$$\frac{\partial f^*}{\partial \alpha_j} = f_1 \frac{\partial x_1^*}{\partial \alpha_j} + f_2 \frac{\partial x_2^*}{\partial \alpha_j} + \cdots + f_n \frac{\partial x_n^*}{\partial \alpha_j} + \frac{\partial f}{\partial \alpha_j}$$

$$-\lambda g_1 \frac{\partial x_1^*}{\partial \alpha_j} - \lambda g_2 \frac{\partial x_2^*}{\partial \alpha_j} - \cdots - \lambda g_n \frac{\partial x_n^*}{\partial \alpha_j} - \lambda \frac{\partial g}{\partial \alpha_j}$$

Now collect up terms. We have:

$$\frac{\partial f^*}{\partial \alpha_j} = (f_1 - \lambda g_1) \frac{\partial x_1^*}{\partial \alpha_j} + (f_2 - \lambda g_2) \frac{\partial x_2^*}{\partial \alpha_j}$$

$$\cdots + (f_n - \lambda g_n) \frac{\partial x_n^*}{\partial \alpha_j} + \frac{\partial f}{\partial \alpha_j} - \lambda \frac{\partial g}{\partial \alpha_j}$$

and we see that from the FOCs each of the terms of the form:

$$f_i - \lambda g_i$$

is zero at $(x^*, \lambda^*)$. We conclude that at a stationary point:

$$\frac{\partial f^*}{\partial \alpha_j} = \frac{\partial f}{\partial \alpha_j} - \lambda^* \frac{\partial g}{\partial \alpha_j}$$

where any of the augmented choice variables appearing on the right-hand side are evaluated at the optimal choices. This is the Envelope Theorem for equality-constrained optimization. Using the Lagrangian, we can write down the result in an even more compressed form:

$$\frac{\partial f^*}{\partial \alpha_j} = \frac{\partial L}{\partial \alpha_j}$$

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7.10 Interpretation of the Lagrange multiplier

Consider the problem
\[
\max / \min_x f(x; \alpha) \\
s.t. \quad g(x; \alpha) = b
\]
for which the Lagrangian is
\[
\mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x; \alpha))
\]

Let \((x^*(\alpha), \lambda^*(\alpha))\) be a stationary point of the Lagrangian and consider the quantity:
\[
\frac{\partial f^*(\alpha)}{\partial b}.
\]

By the Envelope Theorem we can immediately write down that:
\[
\frac{\partial f^*(\alpha)}{\partial b} = \frac{\partial \mathcal{L}}{\partial b} = \lambda^*
\]

In other words, the Lagrange multiplier represents the change in the value of the objective function obtained by relaxing the constraint \(g(x; \alpha) = b\) by one unit.

Let’s pursue this for a moment. In section 6.3 we considered the unconstrained optimization problem
\[
f(x_1, x_2) = x_1^2 + x_2^2 + 0.5x_1x_2 + 4x_2
\]
and we found that the solution was:
\[
x_1^* = \frac{8}{15}, \quad x_2^* = \frac{-32}{15}.
\]

Now let’s add a constraint to this problem:
\[
x_1 - x_2 = \frac{40}{15}
\]
Take a minute to look at this: what do you think is going on?

The Lagrangian is:
\[
\mathcal{L} = x_1^2 + x_2^2 + 0.5x_1x_2 + 4x_2 + \lambda \left(\frac{40}{15} - x_1 + x_2\right)
\]
using our standard procedure of writing the constraint, with the left-hand side (here, 40/15) starting off, and the remainder subtracted from it, leading to the FOCs:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 0 : 2x_1 + 0.5x_2 - \lambda = 0 \\
\frac{\partial L}{\partial x_1} &= 0 : 2x_2 + 0.5x_1 + 4 - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 0 : \frac{40}{15} - x_1 + x_2 = 0
\end{align*}
\]

From the first two FOCs, eliminating \(\lambda\),

\[
\begin{align*}
2x_1 - 0.5x_2 &= -2x_2 - 0.5x_1 - 4 \\
2\frac{1}{2}x_1 &= -2\frac{1}{2}x_2 - 4 \\
x_1 &= -x_2 - \frac{8}{5}
\end{align*}
\]

and then substituting into the third:

\[
\begin{align*}
x_1 - x_2 &= \frac{40}{15} \\
-x_2 - \frac{8}{5} - x_2 &= \frac{40}{15} \\
-2x_2 &= \frac{40}{15} + \frac{8}{5} = \frac{64}{15}
\end{align*}
\]

So we have found that:

\[
\begin{align*}
x_2 &= \frac{-32}{15} \\
x_1 &= \frac{8}{15}
\end{align*}
\]

But what about \(\lambda^*\)? From the first FOC, we have:

\[
\begin{align*}
\lambda &= 2x_1 - 0.5x_2 \\
    &= \frac{16}{15} - \frac{16}{15} \\
    &= 0
\end{align*}
\]

What does all this mean? The constrained problem has the same solution as the unconstrained one: in other words, the constraint was redundant (as you can see if
you look at it carefully) and the solution to the Lagrangian told us that by finding
that $\lambda^* = 0$. In other words, when the optimal value of the Lagrange multiplier is
zero, the constraint isn’t important. Of course, in an equality constrained problem
this is unlikely to occur — look at the steps we had to go through to construct
an example where the constraint didn’t matter — but it could be important when
we look at optimization subject to inequality constraints, where, given a constraint
of the form $g(x; a) \leq 0$ the optimal value of $x$ might well be consistent with
$g(x^*; a) < 0$ — ie the constraint is not binding.

7.11 Multiple equality constraints

Consider the problem:

$$\max \ 	ext{or} \ \min_x f(x; a)$$
$$s.t \quad g^1(x; a) = b_1$$
$$\quad g^2(x; a) = b_2$$
$$\quad \vdots$$
$$\quad g^R(x; a) = b_2$$

where we now have $R$ equality constraints. We solve this problem in a precisely
analogous way as we did for the 1-constraint case. For each equality constraint we
introduce a separate Lagrange multiplier, $\lambda_i$ for the $i$-th constraint, and form the
Lagrangian the same way as before:

$$L(x, \lambda; a) = f(x; a) + \lambda_1 (b_1 - g^1(x; a)) + \lambda_2 (b_2 - g^2(x; a)) + \cdots + \lambda_R (b_R - g^R(x; a))$$

and optimize it as an unconstrained problem in the $n + R$ augmented variables
$(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_R)$. We will now have $n + R$ FOCs, one for each of
the $x$’s ($n$ of them) and one for each of the $\lambda$’s ($R$ of them): 

\[
\frac{\mathcal{L}(x, \lambda; \alpha)}{\partial x_i} = 0 \\
\vdots \\
\frac{\mathcal{L}(x, \lambda; \alpha)}{\partial x_n} = 0 \\
\frac{\mathcal{L}(x, \lambda; \alpha)}{\partial \lambda_1} = 0 \\
\vdots \\
\frac{\mathcal{L}(x, \lambda; \alpha)}{\partial \lambda_R} = 0
\]

A stationary point is found by appealing to the Implicit Function Theorem and solving the FOCs simultaneously for the augmented choice variables. When we do so we obtain:

\[
x^*_i(\alpha), \quad i = 1, 2, \ldots, n \\
\lambda^*_j(\alpha), \quad j = 1, 2, \ldots, R
\]

We can formulate the Lagrangian directly in vector terms. Let $h(x; \alpha)$ be a $R \times 1$ column vector whose $i$-th row is $g^i(x; \alpha) - b_i$ and let $\lambda$ be an $R \times 1$ column vector of Lagrange multipliers. Then our Lagrangian can be written compactly as:

\[
\mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda^T (h(x; \alpha))
\]

where $\lambda^T$ is the transpose of the column vector $\lambda$, i.e. a row-vector.

### 7.12 SOC for multiple equality constraints

The bordered Hessian for an $R$-constraint problem will be an $(n + R) \times (n + R)$ square matrix. As we have set it up, the last $R$ rows and columns will be the border, and will represent the first-order partial derivatives of the constraints with respect to the original choice variables (the $x$’s).

As before, a border-preserving principal minor of order $k$ is the determinant of the matrix that remains when we delete $n - k$ rows and the corresponding columns, with the proviso that we do not delete any of the border rows and columns in their entirety.
Since the entire Hessian has \( n + R \) rows and columns, we see that a border-preserving principal minor of order \( k \) will have \( (n + R) - (n - k) = R + k \) rows and columns.

To form the unique leading border-preserving principal minor of order \( k \), we delete the last \( n - k \) non-border rows and the corresponding columns.

To state the SOC for this problem, suppose we have deleted (non-border) rows and columns, and we are looking at an \( m \times m \) matrix. Then the SOC is as follows, where \( R \) is the number of equality constraints:

- **Maximum:** For a maximum, the determinant of an \( m \times m \) border-preserving principal minor must have sign \((-1)^{m-R}, \) for \( m \geq 1 + 2R \).
- **Minimum:** For a minimum, the determinant of an \( m \times m \) border-preserving principal minor must have sign \((-1)^R, \) for \( m \geq 1 + 2R \)
- Otherwise we have a saddle point.

As before, if the unique leading border-preserving principal minors have these properties, then all of them do.

### 7.13 SOCs in practice

Suppose you want to check the SOCs for an \( R \)-constraint problem. As effective way to proceed, using the leading principal minors, is as follows:

- Compute \( S = 1 + 2R \). This is the smallest minor you will need to consider.
- Now, starting from the last non-border column, delete one, then two (etc) rows and the corresponding columns. At each step compute the determinant. Stop when you have reached an \( S \times S \) matrix.
Example: suppose you have 5 independent variables and 2 equality constraints. Then the bordered Hessian will be a $7 \times 7$ matrix, in the following form:

$$H = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & b_{11} & b_{12} \\
    a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & b_{21} & b_{22} \\
    a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & b_{31} & b_{32} \\
    a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & b_{41} & b_{42} \\
    a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & b_{51} & b_{52} \\
    b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & 0 & 0 \\
    b_{12} & b_{22} & b_{32} & b_{42} & b_{52} & 0 & 0
\end{bmatrix}$$

Here the $b$'s are the border.

We now want to generate all the relevant border-preserving leading principal minors of this, whose signs will need to be checked in the SOCs. First, compute the size of the smallest principal minor we will need to consider: in this case $S = 1 + 2R = 1 + 4 = 5$.

The first of the principal minors is simply $H$ itself. Since this is a $7 \times 7$ matrix, the sign of its determinant should be $(-1)^{m-R} = (-1)^{7-2} = (-1)^5 < 0$ for a maximum; and $(-1)^R = (-1)^2 > 0$ for a minimum.

Now start deleting non-border rows and columns from the right of $H$. Since we can’t delete the border, the first deletable row and column is number 5, i.e. the column containing the element $a_{15}$, and the row with $a_{51}$. Deleting this row and column will leave a $6 \times 6$ matrix, and will be a border-preserving leading principal minor of order 4:

$$H_4 = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} & b_{11} & b_{12} \\
    a_{21} & a_{22} & a_{23} & a_{24} & b_{21} & b_{22} \\
    a_{31} & a_{32} & a_{33} & a_{34} & b_{31} & b_{32} \\
    a_{41} & a_{42} & a_{43} & a_{44} & b_{41} & b_{42} \\
    b_{11} & b_{21} & b_{31} & b_{41} & 0 & 0 \\
    b_{12} & b_{22} & b_{32} & b_{42} & 0 & 0
\end{bmatrix}$$

Its sign should be: $(-1)^{m-R} = (-1)^{6-2} = (-1)^4 > 0$ for a maximum; and $(-1)^R = (-1)^3 > 0$ for a minimum.

Since this was greater than the minimum size (5) we can delete another row and column. Working again from the right of $H_4$, the next deletable column is the one containing $a_{14}$, and the row is the one with $a_{41}$. The result will be a $5 \times 5$
matrix, and a border-preserving leading principal minor of order 3:

\[
H_3 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_{11} & b_{12} \\
a_{21} & a_{22} & a_{23} & b_{21} & b_{22} \\
a_{31} & a_{32} & a_{33} & b_{31} & b_{32} \\
b_{11} & b_{21} & b_{31} & 0 & 0 \\
b_{12} & b_{22} & b_{32} & 0 & 0
\end{bmatrix}
\]

For a maximum we require that the sign of this be \((-1)^{5-2} = (-1)^3 < 0\) for a maximum; and \((-1)^2 > 0\) for a minimum. Since this is a 5 \(\times\) 5 matrix and \(S = 5\) we stop here. These are all the conditions that you need to check for this particular problem structure.

### 7.14 Setting up the Lagrangian

Go back to the 1-constraint problem:

\[
\max_{x} \min_{\lambda, \alpha} f(x; \alpha) \\
\text{s.t.} \quad g(x; \alpha) = b
\]

When we discussed this problem, we set up the Lagrangian as:

\[
\mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x; \alpha))
\]

You may ask: why do we have to do it that way? Why couldn’t we subtract the term \(\lambda(b - g(x; \alpha))\); or why couldn’t we formulate the inner term as \(g(x; \alpha) - b\) rather than \(b - g(x; \alpha)\)? There are really two answers to these questions.

First, as far as the optimal \(x\)’s are concerned, it doesn’t matter. The reason is that if \(x^*\) satisfies \(g(x^*; \alpha) - b = 0\) then it obviously satisfies \(b - g(x^*; \alpha) = 0\) too. And if either of these is true then obviously multiplying the left-hand side by either \(\lambda\) or \(-\lambda\) won’t affect this.

However, what the different setups will do is change the sign of the Lagrange multiplier. If we are just focussing on the \(x\)’s, this is unimportant; but as we have seen, the Lagrange multiplier has a potentially interesting interpretation, connected with the change of the indirect objective function \(f^*\) as \(b\) changes. So we usually set things up so that the Lagrange multiplier has this interpretation directly. This means that we can use either of the following formulations:

\[
\mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) + \lambda(b - g(x; \alpha)) \\
\mathcal{L}(x, \lambda; \alpha) = f(x; \alpha) - \lambda(g(x; \alpha) - b)
\]
since in either case, the Envelope Theorem tell us that:
\[
\frac{\partial f^*}{\partial b} = \frac{\partial L}{\partial b} = \lambda^*
\]
whereas if we set up the problem as, say, \( L(x, \lambda; a) = f(x; a) - \lambda(b - g(x; a)) \) then we would have \( \partial f^*/\partial b = -\lambda^* \). There’s nothing wrong with this, but it’s marginally less convenient, because of the minus sign. So we’ll generally choose a formulation such that we get \( \partial f^*/\partial b = \lambda^* \).

### 7.15 The Lagrange Sufficiency Theorem

The Lagrangian approach to the solution of the equality-constrained problems based on the following result, which we state for a minimization problem with one equality constraint:

\[
\min_{x} f(x; a) \\
\text{s.t} \quad g(x; a) = b
\]

We require that \( x \in X \subseteq \mathbb{R}^n \) (ie, \( x \) is an \( n \)-vector). Note that this means that \( x \) must be finite. We define the feasible set \( X_0 \) as \( X_0 = \{ x \in X : g(x; a) = 0 \} \). That is, \( X_0 \) is the set of allowable \( x \)'s which satisfy the constraints.

**Theorem 1** Suppose we can find (finite) numbers \( x^* \in X_0 \) and \( \lambda^* \) such that

\[
\mathcal{L}(x^*, \lambda^*; a) \leq \mathcal{L}(x, \lambda^*; a) \quad \text{for all } x \in X
\]

Then \( f(x^*) \leq f(x) \) for all \( x \in X_0 \). That is, \( x^* \) solves the equality-constrained minimization problem.

**Proof.** Suppose \( x^* \in X_0 \). Then for any \( \lambda \), say \( \lambda^* \),

\[
\mathcal{L}(x^*, \lambda^*; a) = f(x^*; a) + \lambda^*(g(x^*; a)) = f(x^*)
\]

(since \( x \in X_0 \) means that the equality constraints are satisfied, and therefore that \( g(x; a) = 0 \)). Thus

\[
\mathcal{L}(x^*, \lambda^*; a) = f(x^*) \quad \text{for } x^* \in X_0
\]
and therefore

\[ f(x^*) = \mathcal{L}(x^*, \lambda^*; \alpha) \leq \mathcal{L}(x, \lambda^*; \alpha) \quad \text{for all } x \in X \]

\[ = f(x) \quad \text{for } x \in X_0 \]

where the first line uses the hypothesis of the Theorem and the second line restricts \( x \) to values satisfying the constraints, which means that \( \mathcal{L}(x, \lambda^*; \alpha) = f(x) \). Thus, when both \( x^* \) and \( x \) satisfy the constraints,

\[ f(x^*) \leq f(x) \quad \text{for } x^*, x \in X_0 \]

which says that \( x^* \) solves the minimization problem. ■

8 Summary and General Formulation

We can now combine our results for the cases of unconstrained and equality-constrained optimization. We consider a problem of optimizing \( f(x; \alpha) \) subject to \( R \) constraints of the form \( g^i(x; \alpha) = b_i \).

We form the Lagrangian:

\[ \mathcal{L} = f(x; \alpha) + \lambda_1(b_1 - g^1(x; \alpha)) + \lambda_2(b_2 - g^2(x; \alpha)) + \ldots \]

where we understand that if \( R = 0 \), then all the Lagrange multipliers are zero (ie everything after \( f(x; \alpha) \) is omitted).

There will be \( n + R \) separate FOCs:

\[ \frac{\partial \mathcal{L}}{\partial x_i} = 0 : f_i - \sum_j \lambda_j g^j_i = 0 \quad i = 1, 2, \ldots, n \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda_j} = 0 : b_j - g^j(x; \alpha) = 0 \quad j = 1, 2, \ldots, R \]

where \( g^j_i = \frac{\partial g^j(x; \alpha)}{\partial x_i} \). Assuming that the conditions of the Implicit Function Theorem hold, we can solve the FOCs for the (augmented) choice functions \( x^*(\alpha) = x^*_1(\alpha), x^*_2(\alpha), \ldots, x^*_n(\alpha) \) and \( \lambda^*(\alpha) = \lambda^*_1(\alpha), \lambda^*_2(\alpha), \ldots, \lambda^*_R(\alpha) \).

The bordered Hessian will be composed of blocks:

\[
H = \begin{bmatrix}
\mathcal{L}_{xx} & \mathcal{L}_{x\lambda} \\
\mathcal{L}_{x\lambda}^T & 0
\end{bmatrix}
\]
where \( L_{xx} \sim n \times n; L_{x\lambda} \sim n \times R; L_{\lambda\lambda} \sim R \times n \) and \( 0 \) stands for an \( R \times R \) block of zeros. We will want to examine the signs of the determinants of certain \( m \times m \) matrices extracted from the Hessian, which will be the (border-preserving) (leading) principal minors of \( H \). The minimum size of these matrices will be \( S = 1 + 2R \).

Then at a stationary point satisfying the FOCs we have:

- **Maximum**: The sign of an \( m \times m \) matrix must be \((-1)^{m-R}\) for all \( m \geq S\)
- **Minimum**: The sign of an \( m \times m \) matrix must be \((-1)^{R}\) for all \( m \geq S\)
- Otherwise the stationary point is a saddle-point.

The indirect objective function is:

\[
f^*(x^*(\alpha); \alpha)
\]

and the Envelope Theorem says that:

\[
\frac{\partial f^*}{\partial a_j} = \frac{\partial L}{\partial a_j} \text{ evaluated at } (x^*(\alpha), \lambda^*(\alpha))
\]

9 Concavity, Convexity, And All That

9.1 Concavity and quasi-concavity

Suppose we are trying to solve the problem of maximizing a function of a single independent variable \( f(x) \), and we have an \( x_1 \) that satisfies the FOC \( f'(x_1) = 0 \) and the SOC \( f''(x_1) < 0 \). Then \( f \) has a maximum at \( x_1 \). However, since we have only examined the behavior around \( x_1 \), the most we can say is that it is a local maximum. What additional facts would we need in order to conclude that it is also a global maximum?
One obvious suggestion is that $f$ have the upturned shape shown in the figure above (in three dimensions we’d describe it as an overturned bowl). A function whose graph has this shape is said to be **concave**.

There are at least two geometric ways to characterize concavity. We have:

- **Tangent test for concavity:** at any point $x$, $f$ is never above its tangent at that point.

- **Secant test for concavity:** for any two points, $f$ is never below the secant line connecting the two points.

For the $f$ in the figure above, the three points satisfy the tangent test, and the secant between $x_2$ and $x_3$ shows that the secant test is satisfied for this pair of points. You can see that in fact that the tests will be satisfied for any points on this function.

In the figure above, $f$ is not concave: at $x_1$, $f$ is below its tangent, while at $x_2$ it is above the tangent. And considering the secant line connecting $x_1$ and $x_2$, we
see that sometimes \( f \) is above the secant, and sometimes below.

The definition of concavity allows \( f \) to lie on the tangents or on the secant lines. Thus, a straight line is concave. If we want to rule this out, we can insist on a function being strictly concave. We have:

- **Tangent test for strict concavity:** at any point, \( f \) is below its tangent at that point.
- **Secant test for strict concavity:** for any two points, \( f \) is above the secant line connecting the two points.

It is easy to construct an algebraic characterization of the secant tests that is generalizable to higher dimensions. Fix two points \( x_1 \) and \( x_2 \). At \( x_1 \) the value of \( f \) is \( f(x_1) \), and at \( x_2 \) we have \( f(x_2) \). The points on the secant line joining \( f(x_1) \) and \( f(x_2) \) are given by the convex combination \( \theta f(x_1) + (1 - \theta) f(x_2) \), for \( 0 \leq \theta \leq 1 \). And the set of points between \( x_1 \) and \( x_2 \) are given by \( \theta x_1 + (1 - \theta) x_2 \), again for \( 0 \leq \theta \leq 1 \). So for a point \( \theta x_1 + (1 - \theta) x_2 \) between \( x_1 \) and \( x_2 \) we want to compare the value on the secant:

\[
\theta f(x_1) + (1 - \theta) f(x_2)
\]

with the value on \( f \) itself:

\[
f(\theta x_1 + (1 - \theta) x_2)
\]

Then our algebraic version of the secant test is:

- **Concavity:** A function \( f \) is concave if, for any two points \( x_1 \) and \( x_2 \) in the domain of \( f \), we have, for any \( 0 \leq \theta \leq 1 \),

\[
f(x_1 + (1 - \theta) x_2) \geq \theta f(x_1) + (1 - \theta) f(x_2)
\]

- **Strict concavity:** A function \( f \) is strictly concave if, for any two points \( x_1 \) and \( x_2 \) in the domain of \( f \), we have, for any \( 0 \leq \theta \leq 1 \),

\[
f(x_1 + (1 - \theta) x_2) > \theta f(x_1) + (1 - \theta) f(x_2)
\]

(The only difference between the two cases is that in the second, the inequality is strict). Note that it is completely generalizable to the case where \( x_1 \) and \( x_2 \) are vectors and \( f \) is vector-valued.
If $f$ is concave at all points in its domain, then if we have a stationary point $x_1$ with $f'(x_1) = 0$ we can conclude that $f$ has a global maximum at $x_1$. But now consider the figure below.

![Graph of a concave function with stationary points](image)

As drawn, $f$ certainly has a global maximum at $x_1$, but $f$ is not concave, as we can see by applying the secant test to points $x_2$ and $x_3$. So we should be able to supply a weaker condition than concavity in order to guarantee that a stationary point is a global maximum.

One such condition is given by the notion of quasi-concavity. If you stare at the figure for a while, one thing you may notice is that while $f$ isn’t always above a secant line, what is true is that it is always above the lower of the two points forming the secant. For example, while $f$ isn’t above the secant joining $x_2$ and $x_3$, it is certainly above $x_2$, the lower of the two points. This is precisely the idea of quasi-concavity. Algebraically we have:

- **Quasi-concavity:** A function $f$ is quasi-concave if for any two points $x_1$ and $x_2$ in the domain of $f$, we have, for any $0 \leq \theta \leq 1$,
  
  $$f(x_1 + (1 - \theta)x_2) \geq \min\{f(x_1), f(x_2)\}$$

- **Strict quasi-concavity:** A function $f$ is strictly quasi-concave if for any two points $x_1$ and $x_2$ in the domain of $f$, we have, for any $0 \leq \theta \leq 1$,
  
  $$f(x_1 + (1 - \theta)x_2) > \min\{f(x_1), f(x_2)\}$$

Again, the only difference between quasi-concavity and strict quasi-concavity is that in the latter, the inequality is strict. And note again that this is generalizable to the case where $x_1$ and $x_2$ are vectors and $f$ is vector-valued.
9.2 Convexity and quasi-convexity

Convexity is in a sense the mirror image of concavity. If \( f \) is concave and we flip it around a line parallel to the \( x \)-axis, we get a convex function. In three dimensions, a concave function looks like a bowl. Geometrically, we have characterizations of concavity in terms of tangents and secants:

- **Tangent test for convexity:** at any point \( x \), \( f \) is never below its tangent; or
- **Secant test for convexity:** for any two points, \( f \) is never above the secant line connecting the two points.

and we have a similar pair of tests for strict convexity. Note that a straight line is convex but not strictly convex. By our previous results, a straight line is therefore both concave and convex, but neither strictly concave nor strictly convex.

In the figure above, \( f \) is convex (in fact, strictly convex) for all points. If \( x_1 \) is a stationary point of \( f \) satisfying the FOC for a local minimum and if \( f \) is convex over its domain, then \( x_1 \) is a global minimum.

You should be able to construct a picture where \( f \) has a global minimum but is not convex over its domain. For precisely the same reasons as with concavity, we are led to the parallel notion of quasi-convexity. If \( f \) has a local minimum and is also quasi-convex throughout its domain, then we have found a global maximum.
The algebraic definitions, via the secant test, are the same as for concavity, with
the inequalities reversed, and for the quasi versions we look at the max of the two
points rather than the min. Thus we have:

- **Convexity:** A function $f$ is convex if, for any two points $x_1$ and $x_2$ in the
domain of $f$, we have
  \[ f(x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta) f(x_2) \]

- **Strict convexity:** A function $f$ is strictly convex if, for any two points $x_1$
  and $x_2$ in the domain of $f$, we have, for all for $0 \leq \theta \leq 1$,
  \[ f(x_1 + (1 - \theta)x_2) < \theta f(x_1) + (1 - \theta) f(x_2) \]

- **Quasi-convexity:** A function $f$ is quasi-convex if for any two points $x_1$
  and $x_2$ in the domain of $f$, we have, for all for $0 \leq \theta \leq 1$,
  \[ f(x_1 + (1 - \theta)x_2) \leq \max[f(x_1), f(x_2)] \]

- **Strict quasi-convexity:** A function $f$ is strictly quasi-convex if for any
two points $x_1$ and $x_2$ in the domain of $f$, we have, for all for $0 \leq \theta \leq 1$,
  \[ f(x_1 + (1 - \theta)x_2) < \max[f(x_1), f(x_2)] \]

In the figure above, $f$ was strictly quasi-convex at $x_2$ and $x_3$ by the secant test.
Inspection of the figure shows that this holds for any two points on $f$, so since at
$x_1$ we have $f'(x_1) = 0$, $f$ has a global minimum at $x_1$.

### 9.3 Connection with SOCs

There is a connection between these ideas and the second-order conditions for un-
constrained and equality constrained optimization that we have been discussing.
Specifically, the conditions for a function being concave are the SOCs for an un-
constrained maximum, and for convexity those for an unconstrained minimum.
Thus we have:

- **Concavity:** If the principal minors of order $k$ of the Hessian matrix of $f$
have sign $(-1)^k$ then $f$ is concave. Another way of putting this is that the
  principal minors alternate in sign, beginning with the negative.
• **Convexity:** If the principal minors of order $k$ are all positive for all $k$, then $f$ is convex.

A similar result holds for the leading principal minors. If you have trouble remembering which is which, we would usually be trying to maximize a concave function (overturned bowl) so the condition for a function’s being concave are those for an unconstrained maximum.

Similarly, the conditions for $f$ being quasi-concave or quasi-convex are those for a single equality-constrained problem, that is, conditions on the bordered Hessian, where of course the border is formed by the first partials of $f$ itself. Thus we have:

• **Quasi-concavity:** If the border-preserving principal minors of order $k > 2$ of $f$ have sign $(-1)^k$ then $f$ is quasi-concave.

• **Quasi-convexity:** If the border-preserving principal minors of order $k > 2$ of $f$ have sign $-1$, i.e., are < 0, then $f$ is quasi-convex.

Again, if the leading border-preserving principal minors have these properties, then $f$ has the given curvature.

Finally note that the SOCs for the case of equality-constrained optimization amount to stating that the Lagrangian $L$ is strictly quasi-concave (or convex, as the case may be).

### 10 Inequality Constraints

We will not be using this material in the course: it is presented here only for completeness.

#### 10.1 Maximization problems: setup

The canonical form of a maximization problem with one inequality constraint is:

$$\max_{x} \quad f(x; a)$$

subject to

$$g(x; a) \leq b$$
The logic behind the direction of the constraint is that for a maximization problem we would often want to make $x$ as large as possible, and the direction of the constraint restricts our ability to do so. Of course, we can always convert a constraint of the form $h(x; \alpha) \geq 0$ to the required form by multiplying through by $-1$. Note that we are not imposing nonnegativity constraints on the individual $x$ variables (constraints of the form $x_i \geq 0$): doing this implies additional considerations on the solution, which you can read in many texts, but which I do not consider in these notes.

The essential breakthrough in considering this problem came from Kuhn and Tucker, who suggested that we convert the problem to one with an equality constraint, which we already know how to handle. They did this by adding a slack variable, say $z$, to the problem, reflecting the difference between the left-hand and right-hand sides of the constraint. Since when $g$ is strictly less than zero, it will need to be brought up to zero, we need to add a positive quantity to the left-hand side. The result is the transformed problem in equality form:

$$\max_x f(x; \alpha)$$
$$\text{s.t. } g(x; \alpha) + z^2 = b$$

Note that in some texts you will see the constraint written as $g(x; \alpha) + z = 0$ but that sort of misses the point, because in that case we would have to add that $z$ must be greater than or equal to zero, and the whole point of adding the slack variable was to get rid of the inequality constraint.

### 10.2 Maximization problems: Lagrangian and FOCs

Given our reformulated problem, the natural thing to do is to attack it via the method of Lagrange multipliers. The Lagrangian is:

$$\mathcal{L} = f(x; \alpha) + \lambda (b - g(x; \alpha) - z^2)$$

and we want to maximize this in terms of the (doubly) augmented set of variables $x$, $z$ and $\lambda$. 

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The FOCs are straightforward to obtain:

\[ \frac{\partial L}{\partial x_i} = 0 : f_i - \lambda g_i = 0 \quad i = 1, 2, \ldots, n \]

\[ \frac{\partial L}{\partial z} = 0 : -2\lambda = 0 \]

\[ \frac{\partial L}{\partial \lambda} = 0 : -[g(x; a) + z^2] = 0 \]

Let’s think about these for a bit. First of all, by the Lagrange Sufficiency Theorem we are looking for a finite maximum of \( L \). But if \( \lambda < 0 \) then we can make \( L \) as large as we like by making \( z \) large (positive or negative). This would violate the conditions of the Lagrange Theorem, so we conclude that we must have:

\[ \lambda \geq 0 \]

Since the second FOC is equivalent to \(-2\lambda z^2 = 0 \) (multiply both sides by \( z/2 \)) we can incorporate the third FOC into this and obtain the equivalent condition:

\[ \lambda [g(x; a)] = 0 \]

Finally note that since we can satisfy this constraint when \( g(x; a) = 0 \) so we need to be explicit that we can have \( g(x; a) \leq 0 \).

Putting all this together we have the Kuhn-Tucker necessary conditions for a maximum:

\[ f_i + \lambda g_i = 0 \quad i = 1, 2, \ldots, n \]

\[ \lambda [g(x; a)] = 0 \]

\[ \lambda \geq 0 \]

\[ g(x; a) \leq b \]

The important thing about these conditions is that they do not involve the slack variable (\( z \)) at all. Also, note that our result for the sign of \( \lambda \) depended on the way we wrote the Lagrangian. If we wrote it in a different form (see section 7.14) then the logic here could change, and we might have to conclude that \( \lambda \leq 0 \).

### 10.3 Maximization problems: solving the FOCs

A straightforward way to attack the FOCs in the inequality-constrained case is:
Focus first on the equalities. Solve these to find candidate stationary points.

Then check to see if those points satisfy the inequalities among the FOCs.

If we find several candidates satisfying the FOCs then we will need to see which is in fact a maximum. The easiest way to do this is probably to compute the value of the objective function at each candidate point.

Note that this is going to be a bit more complicated than in problems with an equality constraint, because the FOC:

$$\lambda [g(x; \alpha)] = 0$$

can be satisfied as an equality in two ways: (i) when $$\lambda = 0$$ or (ii) when $$g(x; \alpha) = 0$$ (or both). This means that when we focus on the equalities we will need to proceed by considering the two cases, and see what happens.

### 10.4 Maximization problems: example

To see how this plays out, let’s consider a simple inequality-constrained problem:

$$\max_x \quad 6x_1 x_2$$

s.t. $$2x_1 + x_2 \leq 10$$

We begin by adding a slack variable to convert the problem to an equality constrained one:

$$\max_x \quad 6x_1 x_2$$

s.t. $$2x_1 + x_2 + z^2 = 10$$

and then we write down the Lagrangian in our usual form:

$$\mathcal{L} = 6x_1 x_2 + \lambda (10 - 2x_1 - x_2 - z^2)$$

Note that if $$\lambda < 0$$ then we can make this as large as we like, so we know that $$\lambda \geq 0$$. The Kuhn-Tucker conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 : 6x_2 - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 : 6x_1 - \lambda = 0$$

$$0 = \lambda (10 - 2x_1 - x_2)$$

$$\lambda \geq 0$$

$$2x_1 + x_2 \leq 10$$
We now analyze these by considering cases.

**Case 1:** suppose \( \lambda > 0 \). Then from the first two conditions we see that:

\[
\frac{6x_2}{6x_1} = 2 \quad x_2 = 2x_1
\]

and then, given that we have assumed that \( \lambda \neq 0 \), the (original) constraint must hold as an equality in order to satisfy the third condition. So:

\[
2x_1 + x_2 = 10 \\
4x_1 = 10 \\
x_1 = \frac{5}{2} = 2\frac{1}{2} \\
x_2 = 5
\]

Next, we need to check that this satisfies the inequality conditions. From the second equality condition we have

\[
\lambda = 6x_1 \\
= 15
\]

so the condition \( \lambda \geq 0 \) is satisfied, and clearly the condition \( 2x_1 + x_2 \leq 10 \) is too (since we’ve solved this as an equality). So \((x_1, x_2) = (2\frac{1}{2}, 5)\) is a candidate solution to our problem; and for this point we compute that the value of the objective function is \( f^* = 6 \times 2\frac{1}{2} \times 5 = 75 \).

**Case 2:** suppose \( \lambda = 0 \). Then from the first two conditions, \( x_1 = x_2 = 0 \). Clearly this point satisfies the inequality \( 2x_1 + x_2 \leq 10 \), so \((x_1, x_2) = (0, 0)\) is another possible solution satisfying the Kuhn-Tucker conditions. But in this case \( f^* = 0 \); so this can’t be the maximum, given our Case 1 result.

We conclude that our maximum position, subject to the given inequality constraint, is \((x_1^*, x_2^*) = (2\frac{1}{2}, 5)\). For this solution \( \lambda^* = 15 \) and the value of the slack variable is \( z^* = 0 \).

### 10.5 Minimization problems: setup

The standard form of a minimization problem with one inequality constraint is:

\[
\min_x \quad f(x; a) \\
\text{s.t.} \quad g(x; a) \geq b
\]
where the logic of the direction of the constraint is that typically in a minimization problem we will be trying to make $x$ as small as possible, and the constraint restricts our ability to do so. Of course, if we are given a constraint of the form $h(x; a) \leq b$ we can always convert it to the required form by multiplying both sides by $-1$.

### 10.6 Minimization problems: Lagrangian and FOCs

The Kuhn-Tucker approach here is precisely analogous to the maximization case. We convert the inequality-constrained problem to an equality-constrained one by subtracting a positive slack variable, resulting in:

$$
\begin{align*}
\min_x & \quad f(x; a) \\
\text{s.t.} & \quad g(x; a) - z^2 = b
\end{align*}
$$

The Lagrangian is:

$$
\mathcal{L} = f(x; a) + \lambda(b - g(x; a) + z^2)
$$

and we want to maximize this in terms of the (doubly) augmented set of variables $x, z$ and $\lambda$. The FOCs are:

$$
\begin{align*}
\frac{\partial \mathcal{L}}{\partial x_i} &= 0 : f_i - \lambda g_i = 0 \quad i = 1, 2, \ldots, n \\
\frac{\partial \mathcal{L}}{\partial z} &= 0 : 2\lambda z = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0 : -[g(x; a) + z^2] = 0
\end{align*}
$$

Thinking about the Lagrange multiplier, we see that if $\lambda < 0$ then we can make $\mathcal{L}$ as small as we please by making $z$ large in absolute value, which would violate the conditions of the Lagrange Sufficiency Theorem. So we conclude that we must have $\lambda \geq 0$. (Note once more that this is a feature of the way we set up the Lagrangian: if we did it differently the logic here could change). The the Kuhn-Tucker necessary conditions for a minimum are then:

$$
\begin{align*}
f_i - \lambda g_i &= 0 \quad i = 1, 2, \ldots, n \\
\lambda [g(x; a)] &= 0 \\
\lambda &\geq 0 \\
g(x; a) &\geq b
\end{align*}
$$
and we would proceed to analyze these just as we did in the maximization case.

11 References

There are many textbooks on optimization, and you may have your own preferences. Here are some that I found useful. The book by Silberberg contains brief reviews of calculus and matrix algebra relevant to optimization, and provides the basic reading for our application of optimization to questions of comparative statistics.


Morton I. Kamien and Nancy L. Schwartz. *Dynamic Optimization*. North-Holland, New York, N.Y., 1981. We will not be doing anything involving dynamic optimization, but Appendix A, sections 1–6 contains useful material on the static optimization questions considered in these notes.