1 Introduction

Welfare economics studies the interplay between two kinds of outcomes at a global (economy-wide) level: outcomes that result from the behavior of individuals, and
outcomes that are interesting in their own right, independently of behavior. In particular, we focus on outcomes that result from individuals’ optimizing and price-taking behavior (the competitive equilibria) and a set of outcomes that we think are desirable, independently of behavior: the Pareto-optimal (or Pareto-efficient, perhaps a less morally loaded term) outcomes. A central result is that under certain conditions the two sets of outcomes coincide: a competitive equilibrium is a Pareto optimum. This is the so-called Second Theorem of Welfare Economics (the first theorem characterizes the existence of a competitive equilibrium). Given this correspondence, it is also of interest to understand the conditions under which it does not hold: when will a competitive equilibrium not be a Pareto-optimum?

There are two approaches to proving results in this field. The calculus-based approach exhibits the implied optimization problems and studies their solutions directly. A limitation of the calculus approach is that it is not well-suited to problems with large numbers of individuals or goods. On the other hand, it is often a more useful way to set up applied problems. The algebraic approach gets round the limitations on large numbers by switching to the algebraic properties of vectors, and is particularly suited to the study of certain existence questions. In these notes, we will make use of both.

1.1 Basic ideas

Here is a rough description of the basic ideas that we will be studying. They will be made more precise in later sections.

1.1.1 Allocations

- An allocation is simply a list of who gets what: an assignment of goods and services (including ownership of resources) to individuals.

- A feasible allocation is one that can be produced given the resources and production technology available to the economy.

1.1.2 Pareto-optimal Allocations

As mentioned above, we focus on a set of allocations that are thought to be desirable independently of behavior. In particular, we single out the Pareto-optimal
A (feasible) allocation is weakly Pareto-optimal if there is no alternative (feasible) allocation such that everyone prefers the alternative to the original. In other words, if an allocation is not weakly Pareto-optimal, there is a (feasible) way to make everyone better off.

A (feasible) allocation is strongly Pareto-optimal if there is no alternative (feasible) allocation such that at least one person prefers the alternative, and everyone else is indifferent. In other words, if an allocation is not strongly Pareto-optimal then there is a (feasible) way to make (at least) one person better off, without making anyone else worse off.

We now show that under some plausible conditions, the two characterizations are the same.

1. First, a strongly Pareto-optimal allocation is always weakly Pareto-optimal. For if an allocation is strongly Pareto-optimal, you can’t make one person better off without hurting someone else. So, obviously, you can’t make everyone better off without hurting someone else.

2. Next, we show that if everyone’s preferences are continuous and strongly monotonic (ie everyone prefers more of every good to less) then a weakly Pareto-optimal allocation is strongly Pareto-optimal. To show this, we prove the logically equivalent result: if an allocation is not strongly Pareto-optimal, then it is not weakly Pareto-optimal.

So suppose an allocation is not strongly Pareto-optimal. Then we can make one person better off without hurting anyone else. This will have to be done by giving this person more of some goods/services. Take that increment and distribute almost all of it to everyone else: we can do this because of continuity. Then we have made everyone else better off (because of strong monotonicity). So the original allocation cannot be weakly Pareto-optimal.

In the remainder of these notes, we will assume that the two characterizations coincide, so that we can use whichever is easier.

It is important to realize that the notion of a Pareto-optimal allocation is completely independent of any mechanism that might give rise to it: it is a characterization only of allocations. There is no reason that we cannot ask whether the Stalinist
allocation (the one resulting when Stalin is in command) is or is not Pareto-optimal; the same applies to any other mechanism that generates allocations.

It is also important to be clear that Pareto-optimality is (ethically) a very weak criterion. To see this, consider a world in which everyone’s preferences are strongly monotonic. Now consider the (feasible) allocation in which I have everything and everyone else has nothing. This allocation is Pareto-optimal, since the only way I can make anyone better off is to give up something myself — and this would make me worse off. But few of us would consider this allocation morally compelling. In the words of Amartya Sen (Collective Choice and Social Welfare, p. 22), an allocation “can be Pareto-optimal and still be perfectly disgusting”.

One way to think about this is to observe that if an allocation is not Pareto-optimal, then there is a free lunch: a way of making at least one person better off without harming anyone else. In that sense, the non-Pareto-optimal allocations are not interesting. But that still leaves open the question of which Pareto-optimal allocation we should aim for. Note that the Pareto standard provides no way to compare different Pareto-optimal allocations: two Pareto-optimal allocations are Pareto-incomparable, and in order to distinguish between them we need another criterion. We will return to this issue in later discussions.

1.1.3 Competitive Equilibria

The second set of allocations that we will be studying result from optimizing, price-taking behavior.

- A (feasible) allocation is a competitive equilibrium if (1) it results from maximizing and price-taking behavior by all individuals; and (2) all markets clear (ie that aggregate supply equals aggregate demand in each market).

It is often easiest to think about a competitive equilibrium as being a price vector (one price for each good/service/resource) such that individuals take those prices as given, maximize, and then the result is one in which all markets clear. An important question here is when such a equilibrium price vector exists at all.
2 The Pure Exchange (Walrasian) Economy

The pure exchange economy is one in which there is no production: there are fixed supplies of all the goods and services (and we don’t ask where they came from). The problem in the pure exchange economy is to allocate those fixed supplies to the individuals in the economy.

To study this, we need some notation.

- There are $I$ individuals and $N$ goods/services.
- Individual $i$’s consumption of the $j$-th good (or service) is written as $x_{ij}$. In general, the first subscript identifies an individual, and the second identifies a good.
- Individual $i$’s consumption bundle is $x^i = (x_{i1}, x_{i2}, \ldots, x_{iN})$.
- Total consumption of good $j$, $x_j$, is the sum of each individual’s consumption of good $j$: $x_j = \sum_i x_{ij}$.
- The fixed total amount of good $j$ available to the economy is $\bar{x}_j$.
- An allocation is a list of who gets what: it is the set of all individual consumption bundles: $x = (x^1, x^2, \ldots, x^I)$.
- An allocation is feasible if total demand for each good $j$ does not exceed the fixed total supply: $x_j = \sum_i x_{ij} \leq \bar{x}_j$, for all $j$.
- Individual $i$ has an initial endowment $\tilde{x}_{ij}$ of good $j$. This is given. We assume that all fixed supplies are allocated to the individuals, so that $\sum_i \tilde{x}_{ij} = \bar{x}_j$, for each good $j$.

2.1 Pareto optimality

We turn now to the Pareto-optimal allocations. We shall make several additional assumptions: we assume that everyone’s preferences are rational, strongly monotonic, and continuous. Thus preferences can be represented by a continuous utility function. Individual $i$’s utility function is $u^i(x^i) = u^i(x_{i1}, x_{i2}, \ldots, x_{iN})$. 
In this setting, we can find a Pareto-optimal allocation as follows: fix the utilities of everyone except (say) the first individual. Then maximize the utility of this individual, subject to the constraint that everyone else gets the pre-determined utility level. It should be clear (a) that the resulting allocation is Pareto-optimal, and (b) that there will be a Pareto-optimal allocation corresponding to each assignment of utilities to the “everyone else” group, so that the set of Pareto-optimal allocations is not unique; and (c) given strong monotonicity, there will no goods/services “left over” in a Pareto-optimal allocation: in the notation of the last sub-section, in a Pareto-Optimal allocation we will have \( x_j = \bar{x}_j \) for all goods \( j \). (Note that this modifies a point in the previous sub-section where we required only that \( x_j \leq \bar{x}_j \)).

In order not to get bogged down in the notation, we will consider a special case of the pure-exchange economy in which there are just two individuals \( I = 2 \) and two goods \( N = 2 \). It should be clear how this could be extended to more goods and/or individuals, though the calculus formalism will become cumbersome to write out.

In the \( 2 \times 2 \) case, the Pareto-optimum is an allocation that maximizes individual 1’s utility, subject to the constraints that (i) individual 2’s utility is some fixed \( \bar{u}_2 \) and (ii) the result is feasible, ie that \( \sum_j x_{ij} = \bar{x}_{ij} \), for all goods \( j \). In other words, our problem is to find an allocation \( x = (x_{11}, x_{12}, x_{21}, x_{22}) \) that solves the problem:

\[
\begin{align*}
\max & \quad u^1(x_{11}, x_{12}) \\
\text{s.t.} & \quad u^2(x_{21}, x_{22}) = \bar{u}_2 \\
& \quad x_{11} + x_{21} = \bar{x}_1 \\
& \quad x_{12} + x_{22} = \bar{x}_2
\end{align*}
\]

where the objective function is to maximize individual 1’s utility subject to the constraints that (i) individual 2’s utility is \( \bar{u}_2 \); (ii) the total amount of good 1 allocated to the two individuals \( (x_{11} + x_{21}) \) uses up the total supply \( (\bar{x}_1) \) of good 1, and (iii) the same holds for good 2. The Lagrangian for this problem is:

\[
\max L = u^1(x_{11}, x_{12}) + \lambda_1 (\bar{u}_2 - u^2(x_{21}, x_{22})) + \lambda_2 (\bar{x}_1 - x_{11} - x_{21}) + \lambda_3 (\bar{x}_2 - x_{12} - x_{22})
\]

We can simplify this by noting that if individual 1 gets \( x_{1j} \) of good \( j \), then individual 2 must get what’s left, ie \( x_{2j} = \bar{x}_j - x_{1j} \). Thus, we can eliminate the two adding-up constraints and write the Lagrangian for the Pareto-optimality problem as one of choosing an allocation \( x_{11}, x_{12} \) for individual 1 only, to solve:

\[
\max L = u^1(x_{11}, x_{12}) + \lambda (\bar{u}_2 - u^2(x_{11}, x_{12}))
\]
which seems less intimidating.

The FOCs for the second version of the problem are:

\[
\frac{\partial L}{\partial x_{11}} = 0 : \frac{\partial u^1}{\partial x_{11}} + \lambda \frac{\partial u^2}{\partial x_{21}} = 0
\]

\[
\frac{\partial L}{\partial x_{12}} = 0 : \frac{\partial u^1}{\partial x_{12}} + \lambda \frac{\partial u^2}{\partial x_{22}} = 0
\]

\[
\frac{\partial L}{\partial \lambda} = 0 : \bar{u}^2 - u^2(x_1 - x_{11}, x_2 - x_{12}) = 0
\]

The first two conditions imply:

\[
\frac{\partial u^1/\partial x_{11}}{\partial u^1/\partial x_{12}} = \frac{\partial u^2/\partial x_{21}}{\partial u^2/\partial x_{22}}
\]

which is readily interpreted (together with the third FOC) as saying that in a Pareto-optimum, individual 1’s indifference curve must be tangent to the given indifference curve \((\bar{u}^2)\) of individual 2.

### 2.2 Price-taking behavior

We now assume that there exist (positive) prices \(p = (p_1, p_2)\) (for our two-good economy) which each individual takes as given. Instead of an exogenous income, individual \(i\) has a beginning endowment \(\tilde{x}^i = (\tilde{x}_{1i}, \tilde{x}_{2i})\) of the two goods, which he or she can dispose of (sell) at market prices. This endowment gives rise to the individual’s budget constraint: any feasible consumption bundle \(x^i\) for individual \(i\) must satisfy \(\sum_j p_j x_{ij} = \sum_j p_j \tilde{x}_{ij}\), with equality since strong monotonicity holds.

Individual \(i\)’s problem in the 2-good economy is to maximize utility subject to his or her budget constraint:

\[
\text{max } u^i(x_{1i}, x_{2i})
\]

s.t. \(p_1 x_{1i} + p_2 x_{2i} = p_1 \tilde{x}_{1i} + p_2 \tilde{x}_{2i}\)

We can make this appear like our previous model of individual behavior if we write \(y^i = p_1 \tilde{x}_{1i} + p_2 \tilde{x}_{2i}\) (\(y^i\) is individual \(i\)’s income, which we previously wrote as \(M_i\)). But there is a fundamental difference. In the earlier work, income was an independent parameter. Here it is not: it is simply the valuation at prices \(p\) of a fixed endowment. One important consequence is that while previously the
Marshallian demands were homogeneous of degree 0 in prices and income, here they are homogeneous of degree 0 in prices only.

Using our new notation, we can write the individual's problem as:

\[
\max_{x_{i1}, x_{i2}} u^i(x_{i1}, x_{i2})
\quad \text{s.t.} \quad p_1 x_{i1} + p_2 x_{i2} = y^i
\]

The Lagrangian is:

\[
L^i = u^i(x_{i1}, x_{i2}) + \mu (y^i - p_1 x_{i1} - p_2 x_{i2})
\]

and we get the familiar FOCs:

\[
\frac{\partial L}{\partial x_{i1}} = 0 : \quad \frac{\partial u^i}{\partial x_{i1}} - \mu p_1 = 0
\]

\[
\frac{\partial L}{\partial x_{i2}} = 0 : \quad \frac{\partial u^i}{\partial x_{i2}} - \mu p_2 = 0
\]

\[
\frac{\partial L}{\partial \mu} = 0 : \quad p_1 x_{i1} + p_2 x_{i2} = y^i
\]

and from the first two FOCs:

\[
\frac{\partial u^i}{\partial x_{i1}} = \frac{p_1}{p_2} \frac{\partial u^i}{\partial x_{i2}}
\]

At an (individual) optimum satisfying the budget constraint, the MRS equals the price ratio. We assume that the SOC is satisfied, so that this represents and interior solution (which amounts to assuming that the utility function is convex).

In a competitive equilibrium for our two person-economy each individual takes the same prices as given, so we get:

\[
\frac{\partial u^i}{\partial x_{i1}} / \frac{\partial u^i}{\partial x_{i2}} = \frac{p_1}{p_2} = \frac{\partial u^j}{\partial x_{j1}} / \frac{\partial u^j}{\partial x_{j2}}
\]

so:

\[
\frac{\partial u^i}{\partial x_{i1}} / \frac{\partial u^i}{\partial x_{i2}} = \frac{\partial u^j}{\partial x_{j1}} / \frac{\partial u^j}{\partial x_{j2}}
\]

We immediately see that this condition is precisely the one (equation (1)) that characterizes a Pareto-optimum. But we need to be a bit careful here. We have examined the result of price-taking optimizing behavior by our two individuals.
We have not worried about whether the individual demands for the two goods are consistent with the fixed total supplies. In other words, we don’t know if there are prices that will lead to this behavior being realized as an equilibrium. So our conclusion here is just: if a competitive equilibrium exists, then it is a Pareto-optimum. In fact, as we will see in later notes, this calculus-based approach is most useful when the two sets of conditions do not coincide: then we will be able to conclude that the result of price-taking behavior is not Pareto-optimal.

But that still leaves open the question of whether there is a competitive equilibrium at all. To study that question, we turn to the algebraic approach.

2.3 Walrasian equilibria

We now work in the more general setting of an $I$-individual and $N$-good economy. The notion of a Walrasian equilibrium is another formalization of the notion of a competitive equilibrium for the pure exchange economy. We say:

- An allocation $x$ is **feasible** if:

$$x_j = \tilde{x}_j \quad \text{for all goods } j$$

ie if total demand for each good $j$ by all individuals equals the total (fixed) endowment of that good. In other words, each market clears.

This makes use of the assumption that all total supplies (the $\tilde{x}$) are part of someone’s endowment (the $\tilde{\tilde{x}}$). The effect is that we can consider only the endowment and don’t need to worry about $\tilde{x}$. However, without this assumption (ie if some good are inititally un-owned) we would need to be more careful about the accounting.

- An allocation $x^i$ **maximizes** $i$’s utility subject to a budget constraint at prices $p$ if, whenever another allocation $x^{i'}$ is preferred by $i$ to $x^i$ we have

$$px^{i'} > p\tilde{x}^i$$

in other words, $x^i$ maximizes $i$’s utility subject to the budget constraint at prices $p$ when any allocation that $i$ prefers to $x^i$ is unaffordable given $i$’s endowment $\tilde{x}_i$ (at prices $p$).

- We now define a **Walrasian equilibrium** as a price vector $p$ (an $N$-vector) and an allocation $x(p) = (x^1(p), \ldots x^I(p))$ such that:
1. \( x(p) \) is feasible (ie all markets clear).
2. For all individuals \( i \), \( x^i(p) \) maximizes \( i \)'s utility subject to the budget constraint at prices \( p \).

## 2.4 Walrasian equilibria and Pareto-optimality

Since the idea of a Walrasian equilibrium is just another way of formulating the pure exchange economy, it should come as no surprise that a Walrasian equilibrium is Pareto-optimal. But let’s prove this anyway, using our new concepts.

- **Theorem** If \((x, p)\) is a Walrasian equilibrium, then it is Pareto-optimal.

  **Proof:** Suppose \( x \) is not Pareto-optimal. Then (using the notion of a weak Pareto-optimum) there is an alternative feasible allocation \( x' \) that everyone prefers to \( x \). So, by the definition of a Walrasian equilibrium, for each person \( i \), we have that \( \sum_j p_j x'_{ij} > \sum_j p_j \tilde{x}_{ij} \) for each \( i \). Now sum this over all individuals: we get:

  \[
  \sum_i \sum_j p_j x'_{ij} > \sum_i \sum_j p_j \tilde{x}_{ij}
  \]

  and interchange the order of summation:

  \[
  \sum_j \sum_i p_j x'_{ij} > \sum_j \sum_i p_j \tilde{x}_{ij}
  \]

  or

  \[
  \sum_j p_j \sum_i x'_{ij} > \sum_j p_j \sum_i \tilde{x}_{ij}
  \]

  But since \( x' \) is (also) feasible, total demand for each good in \( x' \) equals total endowment of that good, or

  \[
  \sum_i x'_{ij} = \sum_i \tilde{x}_{ij}
  \]

  for each good \( j \). Then, inserting this into the left hand side of the previous inequality we have that:

  \[
  \sum_j p_j \sum_i \tilde{x}_{ij} > \sum_j p_j \sum_i \tilde{x}_{ij}
  \]

  which is a contradiction (the left and right sides are exactly the same, so they must be equal, not unequal).
Finally, note that this has made no serious progress over our previous result: all we have shown is that if a Walrasian equilibrium exists, then it is Pareto-optimal. (however, we have done so without appealing to just two individuals or goods).

2.5 Excess demand

We turn now to the real question, whether there exists an equilibrium price vector at all for the pure exchange economy. Write $x_{ij}(p)$ for individual $i$’s Marshallian demand for good $j$ at prices $p$.\(^1\) We define individual $i$’s excess demand for good $j$ (ie his/her demand for $j$ over and above endowment) as:

$$z_{ij}(p) = x_{ij}(p) - \tilde{x}_{ij}$$

The aggregate excess demand for good $j$ is just the sum of the individual excess demands:

$$z_j(p) = \sum_i (x_{ij}(p) - \tilde{x}_{ij})$$

and the aggregate excess demand vector (at prices $p$) for the entire economy is the $N$-vector:

$$z(p) = (z_1(p), z_2(p), \ldots, z_N(p)).$$

The reason for introducing the notion of excess demand is that the market-clearing condition for good $j$ is $z_j(p) = 0$, and for the entire economy market-clearing (ie, feasibility) is $z(p) = 0$, where 0 is an $N$-vector of 0’s.

We note three properties of the excess demand vector:

- $z_j(p)$ is continuous in $p$. This follows from the continuity of utility functions, which implies that the individual Marshallian demand functions $x_{ij}(p)$ are also continuous.
- $z_j(p)$ is homogeneous of degree 0 in $p$.
- \textbf{Walras’ Law} $p \cdot z(p) = 0$. This says that the value of the aggregate excess demand function is always zero.

\(^1\)In the individual behavior notes, we wrote the Marshallian demands as $x'_{ij}(p)$, but we’re trying to avoid clutter wherever possible.
Proof: For each individual $i$, the Marshallian demands satisfy $i$’s budget constraint with equality, so:

$$\sum_j p_j x_{ij} - \sum_j p_j \tilde{x}_{ij} = 0$$
$$\sum_j (p_j x_{ij} - p_j \tilde{x}_{ij}) = 0.$$  

Now sum over all individuals $i$:

$$\sum_i \sum_j (p_j x_{ij} - p_j \tilde{x}_{ij}) = 0$$

and interchange the order of summation:

$$\sum_j \sum_i (p_j x_{ij} - p_j \tilde{x}_{ij}) = 0$$

so that:

$$\sum_j p_j \sum_i (x_{ij} - \tilde{x}_{ij}) = 0$$

or:

$$\sum_j p_j z_j(p) = 0$$

which is what we wanted to show.

To illustrate what Walras’ Law implies, consider our 2-individual, 2-good economy. First, suppose that $z_1(p) < 0$, so that good 1 is in excess supply. Then Walras’ Law implies that we must have $z_2(p) > 0$, ie that good 2 is in excess demand. Second, suppose $z_1(p) = 0$. Then Walras’ Law implies that $z_2(p) = 0$ also. More generally, in an $N$-good economy, if $N - 1$ markets are in equilibrium (ie with $z_j(p) = 0$) then the last one must be, too.

### 2.6 Brouwer’s fixed-point theorem

In order to prove the existence of an equilibrium price vector, we shall need a result from mathematics, known as Brouwer’s fixed-point theorem.

- **Definition**: the $(N - 1)$-dimensional **unit simplex** is the set of positive $N$-vectors $x$ such that $\sum_j x_j = 1$. 

Theorem (Brouwer): if \( f \) is a vector-valued continuous function mapping the \((N - 1)\)-dimensional unit simplex onto itself, then there is a point \( x \) such that \( f(x) = x \). The point \( x \) is called a fixed point of \( f \).

Here’s a sketch of a proof for the special case \( N = 2 \). In this case the unit simplex is the closed unit interval \([0, 1]\).\(^2\) Let \( f \) be a continuous function mapping the closed unit interval \([0, 1]\) onto itself. We want to show that there is an \( x \) such that \( f(x) = x \).

Consider the function \( g(x) = f(x) - x \). This is obviously continuous, and if \( x \) is a fixed point of \( f \) then it satisfies \( g(x) = 0 \). So finding a fixed point of \( f \) is equivalent to finding a zero of \( g(x) \).

Now \( g(0) = f(0) - 0 = f(0) \geq 0 \) since the range of \( f \) is non-negative (the unit interval). And \( g(1) = f(1) - 1 \). This must be non-positive, since the maximum value that \( f \) can take is 1. If \( g(0) = 0 \) or \( g(1) = 0 \) we are done, so assume \( g(0) > 0 \) and \( g(1) < 0 \). In this case, \( g \) goes continuously from a positive value at \( x = 0 \) to a negative value at \( x = 1 \). Therefore (by the intermediate value theorem) there is a point \( x \) where \( g(x) = 0 \). So \( f \) has (at least one) fixed point.

What’s going on can be seen in the picture below. We have a positive \( g(0) \) and a negative \( g(1) \). If you try and join these two points up with a continuous curve (without leaving the unit interval) then obviously you must cross the \( x \)-axis at least once: any point at which you do so is a fixed point of \( f \).

\[^2\text{The closed unit interval is the set of points } 0 \leq x \leq 1, \text{ including the end-points.}\]
2.7 Existence of a Walrasian equilibrium

We are now in a position to prove that there exists a Walrasian (ie utility-maximizing) equilibrium price vector \( p \).\(^3\) Our strategy is (1) to exhibit a certain formula for the prices, a formula that is guaranteed by the Brouwer theorem to be satisfied, ie non-empty; (2) to show that this price formula is a Walrasian equilibrium.

Remember that the results of individual maximization are the Marshallian demands, and that in our Walrasian framework, these are homogeneous of degree 0 in prices. Therefore we can multiply all prices by an arbitrary constant (ie normalize them) and not change anything. If the original price vector is \( p' \) then we shall define the normalized prices by:

\[
p_j = \frac{p_j}{\sum_j p_j'}
\]

(ie we take the arbitrary constant to be \( 1/\sum_j p_j' \)). The result is that the \( p_j \) are positive and sum to 1. So the \( N \)-dimensional vector \( p = (p_1, p_2, \ldots, p_N) \) is in the \( (N-1) \)-dimensional unit simplex.

Consider a vector-valued function \( g(p) \), with \( j \)-th coordinate \( g_j(p) \), defined by

\[
g_j(p) = \frac{p_j + \max(0, z_j(p))}{1 + \sum_k \max(0, z_k(p))}
\]

for \( j = 1, 2, \ldots, N \)

where \( z_j(p) \) is the excess demand for good \( j \) at prices \( p \). The functions \( g_j(p) \) can be thought of as adjustment rules for prices by an omniscient planner: if excess demand for good \( j \) is positive, reduce \( p_j \).

Now, since the maximum function is continuous, \( g_j(p) \) is a continuous function of prices \( p \). Moreover, given positive prices, \( g_j(p) \) is also positive, and for the

\(^3\)In what follows, we assume that all prices are strictly positive. It is straightforward, though it requires a bit more work, to relax this assumption.
sum of the coordinates we have:

\[ \sum_j g_j(p) = \sum_j \frac{p_j + \max(0, z_j(p))}{1 + \sum_k \max(0, z_k(p))} \]

\[ = \frac{1}{1 + \sum_k \max(0, z_k(p))} \sum_j (p_j + \max(0, z_j(p))) \]

\[ = \frac{1}{1 + \sum_k \max(0, z_k(p))} \left( \sum_j p_j + \sum_j \max(0, z_j(p)) \right) \]

\[ = \frac{1}{1 + \sum_k \max(0, z_k(p))} \left( 1 + \sum_j \max(0, z_j(p)) \right) \]

\[ = 1 \]

so that the \( g_j(p) \) are in the unit simplex. Then by Brouwer’s fixed-point theorem, there is a vector \( p^* \) satisfying the fixed-point condition

\[ p_j^* = g_j(p^*) \quad \text{for } j = 1, 2, \ldots, N \]

or

\[ p_j^* = \frac{p_j^* + \max(0, z_j(p^*))}{1 + \sum_k \max(0, z_k(p^*))} \quad \text{for } j = 1, 2, \ldots, N \]

Claim: \((z, p^*)\) is a Walrasian Equilibrium.

Proof: We need to show only that \( z(p^*) = 0 \), since the excess demands already build in utility-maximizing behavior. Multiply out the definition of \( p_j^* \):

\[ p_j^* \left( 1 + \sum_k \max(0, z_k(p^*)) \right) = p_j^* + \max(0, z_j(p^*)) \]

so:

\[ p_j^* \sum_k \max(0, z_k(p^*)) = \max(0, z_j(p^*)) \]

Now multiply both sides by \( z_j(p^*) \):

\[ z_j(p^*) p_j^* \sum_k \max(0, z_k(p^*)) = z_j(p^*) \max(0, z_j(p^*)) \]

and sum over all goods \( j \):

\[ \sum_j z_j(p^*) p_j^* \sum_k \max(0, z_k(p^*)) = \sum_j z_j(p^*) \max(0, z_j(p^*)) \]
By Walras’ Law the term $\sum_j z_j(p^*)p_j^*$ on the left is zero, so

$$0 = \sum_j z_j(p^*) \max(0, z_j(p^*))$$

Now, on the right we have a sum of nonnegative terms, each of which is either 0 or $[z_j(p^*)]^2$, and which must sum to zero. But if one of them was positive, then we’d need a negative term in the sum to balance it out; but we know that our sum contains no negative terms. So the only possibility is that:

$$z_j(p^*) = 0 \quad \text{for } j = 1, 2, \ldots, N$$

which means that our constructed $p^*$ is a price vector giving rise to a Walrasian equilibrium.

### 2.8 The Edgeworth Box

There’s a nice graphical representation of the 2-good 2-person pure exchange economy, known as the Edgeworth Box. In the figure below, the dimensions of the box are the given quantities of the two goods, i.e., $x_1 = x_{11} + x_{21}$ and $x_2 = x_{12} + x_{22}$. Individual 1’s holdings are measured from $0_1$ and individual 2’s from $0_2$. Any point in the box like $\bar{x}$ thus represents an allocation of the total supplies that exactly uses them up — is is a feasible allocation.

The figure below adds in the indifference curves of the two individuals. Individual 1’s curves are measured from $0_1$, have the curvature of the indifference curve $u^1$, and increase in a north-easterly direction. Individual 2’s indifference
curves are measured from $0_2$ and increase (get better) in a southwesterly direction, like $u^2$. As we have seen, the candidates for Pareto-optima are points where the two sets of indifference curves are tangent to one another. This set of all these tangencies is the contract curve, shown in the figure as line $CC$.

What about competitive equilibria? These of course will depend on the individuals’ endowments (because endowments determine income). In the figure below, these are shown as the point $\bar{x} = (\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21}, \bar{x}_{22})$. So a competitive equilibrium price vector is the slope of a line going through $\bar{x}$ that is tangent to a pair of indifference curves (one for each individual), since that is what it would take for both to base their decisions on a common set of prices. In the figure below, the slope of line $PP$ is a price ratio supporting a competitive equilibrium. As you should be able to convince yourself, it is (at least as the figure is drawn) the only possibility for a competitive equilibrium.
But competitive equilibria are not necessarily unique. The figure below shows a case where the preferences of the two individuals — the shapes of their indifference curves — support two competitive equilibria, shown as the slopes of the budget lines PP and QQ.

Can we predict where the two individuals will end up if all they can do is bargain with one another — ie exchange goods and services without a price system? In the figure below, given the endowment (\(\tilde{x}\)), individual 1 is initially on the indifference curve \(\tilde{v}^1\) and individual 2 is on \(\tilde{v}^2\). Clearly, for any trade to be acceptable, individual 1 must get at least utility \(\tilde{v}^1\) and individual 2 must get at least \(\tilde{v}^2\). The result is that all acceptable trades must put the two individuals somewhere in the lens-shaped shaded area. But that’s all that can be said. Note that, where CC is (part of) the contract curve, then there’s no guarantee that the result of individualistic bargaining will be a Pareto-optimum — it could be in the shaded area off the contract curve, depending on the power and/or bargaining skills of the individuals.
This highlights the importance of a competitive price system and justifies a restatement of our result linking competitive behavior and Pareto-optimality:

- **Condition M**: markets (ie prices) exist for all goods.

- **Theorem** (Second Theorem of Welfare Economics): if Condition M holds, then a competitive equilibrium is Pareto-optimal.

We can also see another result. In the figure below, pick any Pareto-optimum (ie any point on the contract curve where the two individuals’ indifference curves are tangent), say $x^*$. Then we can find a budget line PP such that price-taking utility-maximization will result in allocation $x^*$, provided that initial endowments are somewhere along PP.

In other words:

- **Theorem** (Third Theorem of Welfare Economics): Any Pareto-optimum can be realized via competitive utility-maximizing and price-taking behavior by suitably adjusting initial endowments.

Of course, as a practical matter this is of limited usefulness, since it is generally not possible (politically) to simply adjust endowments.
3 The Pure Production Economy

The pure production economy is the production analog of the pure exchange economy: the problem here is how to use resources to produce output, with questions of distribution among the consumers ignored. Specifically:

- There are $M$ resources, $z_1, z_2, \ldots, z_M$. Resource $k$’s total supply is fixed at $\tilde{z}_k$.
- There are (still) $N$ goods. Good $j$ is produced via the production function $x_j = f_j(z_{1j}, z_{2j}, \ldots z_{Mj})$ where $z_{jk}$ is the amount of resource $k$ used in the production of good $j$. For simplicity we will assume that all production functions require strictly positive quantities of all resources. This is somewhat unrealistic, but is not essential to our results.

3.1 Pareto-optimality

Since all goods are “good”, it is clear that the Pareto optimum will be obtained by maximizing the quantity of one (arbitrary) good, while holding the levels of the remaining goods fixed. In the case of the 2-good 2-resource economy this problem is:

$$\max f_1(z_{11}, z_{12})$$

s.t. $\tilde{z}_2 = f_2(z_{21}, z_{22})$

$z_{11} + z_{21} = \tilde{z}_1$

$z_{12} + z_{22} = \tilde{z}_2$

and, if we incorporate the adding-up constraints directly, we get the following Lagrangian in $z_{11}$ and $z_{12}$ only:

$$\mathcal{L} = f_1(z_{11}, z_{12}) + \lambda(\tilde{z}_2 - f_2(\tilde{z}_1 - z_{11}, \tilde{z}_2 - z_{12}))$$

The FOCs are (we assume that the SOCs are satisfied):

$$\frac{\partial \mathcal{L}}{\partial z_{11}} = 0 : \frac{\partial f_1}{\partial z_{11}} + \lambda \frac{\partial f_2}{\partial z_{21}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = 0 : \frac{\partial f_1}{\partial z_{12}} + \lambda \frac{\partial f_2}{\partial z_{22}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 : \tilde{z}_2 - f_2(\tilde{z}_1 - z_{11}, \tilde{z}_2 - z_{12}) = 0$$
and from the first two we see that
\[ \frac{\partial f_1}{\partial z_{11}} = \frac{\partial f_2}{\partial z_{21}} \]
\[ \frac{\partial f_1}{\partial z_{12}} = \frac{\partial f_2}{\partial z_{22}} \]
which says (along with the third FOC) that in a Pareto-optimum the \( x_1 \)-isoquant must be tangent to the \( z_2 \)-isoquant.

### 3.2 Competitive price-taking behavior

Competitive price-taking behavior for firms says that each firm maximizes profits as a price-taker in both the input and output markets. We assume for simplicity that each firm produces just one good, and that firm \( j \) produces (only) good \( x_j \). If good \( x_j \) has market price \( p_j \) and resource \( z_k \) has market price \( r_k \) then the profit-maximization problem of firm \( j \) is
\[
\max \ p_j f_j (z_{j1}, z_{j2}) - r_1 z_{j1} - r_2 z_{j2}
\]
whose FOCs are:
\[
p_j \frac{\partial f_j}{\partial z_{j1}} - r_1 = 0
\]
\[
p_j \frac{\partial f_j}{\partial z_{j2}} = 0
\]
or:
\[
\frac{\partial f_j}{\partial z_{j1}} = \frac{r_1}{r_2}
\frac{\partial f_j}{\partial z_{j2}}
\]
Both firms maximize profits at the same input prices, so we have (for our 2-firm economy):
\[
\frac{\partial f_1}{\partial z_{11}} = \frac{r_1}{r_2} = \frac{\partial f_2}{\partial z_{21}}
\]
so:
\[
\frac{\partial f_1}{\partial z_{12}} = \frac{\partial f_2}{\partial z_{22}}
\]
which is the same at the condition for a Pareto-optimum. We therefore see that if a competitive equilibrium with price-taking behavior exists, then it will be a Pareto-optimum.
3.3 Existence of equilibrium

It should be clear that, with a few notational changes, the pure production economy is structurally exactly the same as the pure exchange economy. Therefore the techniques we used to establish the existence of a Walrasian equilibrium can be used to prove the existence of a resource-price vector that clears the resource markets.