How Efficient is COTA?

An Introduction to DEA

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Transit Operating Subsidies/Trip Since 1970

- US transit operations are increasingly subsidized: operating subsidies have grown from zero in 1970 to more than $2.00 per trip today (or more than 50¢ per trip in real $1970).
- Taxpayers — who ultimately foot the bills — rightly want to know if they are getting value for money from public transit.
- Are transit systems producing as much transit as they can from the resources they have been given, or could they produce transit services with fewer resources?
- This is the question of the efficiency of transit.
- In this lecture I discuss a straightforward and easily implementable approach to this question.
The Feasible Region

- Consider transit systems producing one output ($u_1$) from one input ($x_1$).
- Their production possibilities are shown as the gray area.
- Any input/output combination in the gray area is feasible.
- Combinations outside the gray area are infeasible.

An Observed Transit System

Now consider system 1 ($S_1$) producing output $u_{11}$ from input $x_{11}$.
We can see that System 1 is inefficient, in two separate ways.

Input-Oriented Technical Inefficiency I

- First, it could produce the same output ($u_{11}$) from less input ($\lambda x_{11}$).
- Alternatively: it could contract input from $x_{11}$ to $\lambda x_{11}$ (a factor of $0 < \lambda < 1$) while continuing to produce its observed output.

Input-Oriented Technical Inefficiency II

- We say that System 1 is technically inefficient in an input-oriented sense if there is a factor $\lambda$ ($0 < \lambda < 1$) such that input could be scaled back by $\lambda$ without affecting output.
- Smaller $\lambda$ implies greater inefficiency.
Output-Oriented Technical Inefficiency I

- Second, System 1 could use its observed input ($x_{11}$) but produce more output ($\theta u_{11}$) with it.
- Alternatively: it could expand output from $u_{11}$ to $\theta u_{11}$ (by a factor of $\theta > 1$) while continuing to utilize its observed input.

Output-Oriented Technical Inefficiency II

- We say that System 1 is technically inefficient in an output-oriented sense if there is a factor $\theta > 1$ such that output could be expanded by $\theta$ without needing additional input.
- Larger $\theta$ implies greater inefficiency.

An Efficient System I

- On the other hand, System 2 ($S_2$) is technically efficient, in both senses.
- The only way it can reduce input utilization is by cutting back on output.
- Its smallest possible input-scale-back factor (for given output) is $\lambda = 1$.
- So it is technically efficient in an input-oriented sense.

An Efficient System II

- The only way it can increase output is by utilizing more input.
- Its greatest output expansion factor (for given input) is $\theta = 1$.
- So it is technically efficient in an output-oriented sense.
An Approach

These observations suggest how we might try to evaluate technical efficiency in a given transit system:

- See if it can reduce input while continuing to produce its observed output:
  - If there is no way to do this, then the system is technically efficient in an input-oriented sense.
  - If it *can* reduce input, it is technically inefficient in an input-oriented sense, and the greater the reduction, the more inefficient it is.

- See if it can expand output while continuing to utilize its observed input:
  - If there is no way to do this, then the system is technically efficient in an output-oriented sense.
  - If it *can* expand output, it is technically inefficient in an output-oriented sense, and the greater the expansion, the more inefficient it is.

Before we can use these ideas, we need to face some problems:

- The most serious is that we do not actually observe the feasible region.
- We observe only individual input-output combinations.
- Until we can construct the feasible region, we cannot model inefficiency.

Realism Needed

- Additionally, it is implausible that transit systems produce just one output using just one input.
- In real life we might see:
  - Inputs: vehicles, drivers, maintenance people, oil, fuel . . .
  - Outputs: peak/off-peak, express/local, miles/trips on each mode operated by the transit system.
- We turn now to developing solutions to both these problems.

The Missing Region I

The Best-Practice Frontier

Our first task is to delimit the feasible input-output combinations.

- Obviously, the *observed* input/output combinations are feasible.
- So we need to find a plausible way to construct other feasible but *unobserved* input/output combinations.
- Once this is done, we take the “outer envelope”, giving us the production *frontier*.
- This approach is known as Data Envelopment Analysis, DEA.
- Because it is based on observed data, we call the resulting boundary (envelope) the *Best-Practice Frontier*.
- Note that it will not allow us to say that *all* the observed data is inefficient.
In a 1-input, 1-output world, suppose we observe systems 1 and 2. We regard them as different ways (processes) for utilizing transit inputs to produce transit outputs. We shall assume that all input-output combinations on the line joining the two are also feasible. Intuitively, this says that we can construct a new production process (way of using inputs to produce outputs) by operating the two processes lower intensities and then combining them. That is, we operate process 1 at intensity $\zeta_1$ and process 2 at intensity $\zeta_2$, with $\zeta_1 + \zeta_2 = 1$. In this sense, “mix-n-match”

Mathematically, this says that convex combinations of observed input/output mixes are also feasible. Formally, there exist intensity weights $\zeta_1$ and $\zeta_2$ satisfying $\zeta_1 + \zeta_2 = 1$ such that any inputs and outputs of the form

$u_1 = \zeta_1 u_{11} + \zeta_2 u_{21}$

$x_1 = \zeta_1 x_{11} + \zeta_2 x_{21}$

are also feasible. We can extend this to more than two observations. For each observed system $j$, we define its intensity weight $\zeta_j$.

We then say that any input/output combination satisfying

$u_1 = \zeta_1 u_{11} + \zeta_2 u_{21} + \cdots + \zeta_j u_{j1}$

$x_1 = \zeta_1 x_{11} + \zeta_2 x_{21} + \cdots + \zeta_j x_{j1}$

is also feasible, where $\zeta_1 + \zeta_2 + \cdots + \zeta_j = 1$. 
The set of feasible input/output combinations formed in this way defines the convex hull of the observed data.

• You can visualize this by imagining a lasso around the observed data, and then drawing it tight.

System 1 is producing output $u_{11}$ from input $x_{11}$

• Surely it could produce the same output using more input than $x_{11}$
• So if $(x_{11}, u_{11})$ is feasible, so are $(1.5x_{11}, u_{11}), (2x_{11}, u_{11}), (3x_{11}, u_{11})$ etc.
• We say that inputs are (strongly) disposable.

Conversely, if System 1 uses input $x_{11}$ to produce output $u_{11}$ it could surely use $x_{11}$ to produce less output than $u_{11}$

• So if $(x_{11}, u_{11})$ is feasible, so are $(x_{11}, 0.9u_{11}), (x_{11}, 0.5u_{11}), (x_{11}, 0.26u_{11})$ etc.
• We say that outputs are (strongly) disposable.

Applying these principles to the observed systems $S_1$ and $S_2$ we get the shaded feasible region. The (outer) boundary of the feasible region is the best-practice frontier.
If we also observed $S_3$, note that we’d get a different feasible region (and $S_2$ is now revealed to be inefficient).

We can describe our best-practice frontier formally as follows, given that we observe $J$ single-input / single-output combinations:

The frontier is the outer boundary of the feasible region consisting of:

- All input quantities $x_1$ greater than or equal to ($= $ disposal) all convex combinations of the observed inputs ($= $ mix-n-match or convex hull):
  \[ x_1 \geq \zeta_1 x_{11} + \zeta_2 x_{21} + \cdots + \zeta_J x_{J1} \]
- All output quantities $u_1$ less than or equal to ($= $ disposal) all convex combinations of the observed outputs ($= $ mix-n-match or convex hull):
  \[ u_1 \leq \zeta_1 u_{11} + \zeta_2 u_{21} + \cdots + \zeta_J u_{J1} \]
- The intensities satisfy $\zeta_1 + \zeta_2 + \cdots + \zeta_J$ (for convexity)

We can extend these ideas to more inputs and outputs in the natural way:

- Suppose transit produces $M$ outputs $u_1 \ldots u_M$ from $N$ inputs $x_1 \ldots x_N$
- We then say that our convexity and disposal assumptions hold for each input and output.
- Thus for each input $x_n$ we have a feasible region satisfying
  \[ x_n \geq \zeta_1 x_{1n} + \zeta_2 x_{2n} + \cdots + \zeta_J x_{Jn} \quad n = 1, 2, \ldots, N \]
- And for each output $u_m$ we have a feasible region satisfying
  \[ u_m \leq \zeta_1 u_{1m} + \zeta_2 u_{2m} + \cdots + \zeta_J u_{Jm} \quad m = 1, 2, \ldots, M \]
- In all cases, $\zeta_1 + \zeta_2 + \cdots + \zeta_J$ (for convexity)

We can abbreviate our characterization of the frontier as follows:

- Let $X$ be the $J \times N$ matrix of observed inputs with typical element $x_{jn}$
- Let $U$ be the $J \times M$ matrix of observed outputs with typical element $u_{jm}$
- Let $\zeta$ be a $J \times 1$ column vector of intensity variables for the $J$ observed transit systems: $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_J \geq 0$
- Then our frontier is formed as all input $N$-vectors $x$ and output $M$-vectors $u$ satisfying
  \[ x \geq \zeta' X \]
  \[ u \leq \zeta' U \]
  \[ \zeta' 1 = 1 \]

where the prime denotes transposition and $1$ is a $J$-vector of $1$’s.
Measuring Technical Efficiency

We turn now to measuring efficiency. We focus our attention on system \( j \).

- It is observed to produce the \( M \) outputs \( u_j = u_{j1}, u_{j2}, \ldots, u_{jM} \) (the \( j \)-th row of \( U \))
- It utilizes the \( N \) inputs \( x_j = x_{j1}, x_{j2}, \ldots, x_{jN} \) (the \( j \)-th row of \( X \))

Our question is: is this system efficient or not?

As we have seen, there are two ways we can address this question taking either an input or an output orientation. We focus first on the input orientation.

Measuring Input-Oriented Technical Efficiency

In an input orientation, we ask if \( j \)'s inputs can be scaled back by a proportion \( \lambda \) without leaving the feasible set.

In other words, we seek the smallest \( \lambda \) such that the scaled-back input bundle \( \lambda x_j = \lambda x_{j1}, \lambda x_{j2}, \ldots, \lambda x_{jN} \) is still feasible.

So we need to find the smallest \( \lambda \) (0 < \( \lambda \) < 1) such that:

- For each input \( n \), \( \lambda x_{jn} \geq \zeta_1 x_{1n} + \zeta_2 x_{2n} + \cdots + \zeta_J x_{Jn} \)
- For each output \( m \), \( u_{jm} \leq \zeta_1 u_{1m} + \zeta_2 u_{2m} + \cdots + \zeta_J u_{Jm} \)

A Detailed Example I

We now illustrate this by a detailed example, building up to the optimization linear program on the previous slide.

- We consider the case of \( J = 3 \) observed transit firms
- Each firm produces \( M = 2 \) outputs from \( N = 4 \) inputs
- We focus on system \( j = 3 \) in an input orientation
- So we want to find the smallest \( \lambda \) such that \( j \)'s scaled-back outputs are feasible.
A Detailed Example II

Then we say:

- $j$’s scaled-back input 1 must be feasible (satisfy disposal and convexity) relative to the other systems’ input 1:
  \[ \lambda x_{31} \geq \zeta_1 x_{11} + \zeta_2 x_{21} + \zeta_3 x_{31} \]

- So must $j$’s scaled-back input 2 (relative to the other systems’ input 2):
  \[ \lambda x_{32} \geq \zeta_1 x_{12} + \zeta_2 x_{22} + \zeta_3 x_{32} \]

- And its inputs 3 and 4:
  \[ \begin{align*}
  \lambda x_{33} & \geq \zeta_1 x_{13} + \zeta_2 x_{23} + \zeta_3 x_{33} \\
  \lambda x_{34} & \geq \zeta_1 x_{14} + \zeta_2 x_{24} + \zeta_3 x_{34}
  \end{align*} \]

A Detailed Example III

- $j$’s output 1 must be feasible (satisfy disposal and convexity) relative to the other systems’ output 1:
  \[ u_{31} \leq \zeta_1 u_{11} + \zeta_2 u_{21} + \zeta_3 u_{31} \]

- So must its output 2 (relative to the other systems’ output 2):
  \[ u_{32} \leq \zeta_1 u_{12} + \zeta_2 u_{22} + \zeta_3 u_{32} \]

A Detailed Example IV

So, putting all this together, our problem is to find $\lambda$ and the $\zeta$’s to solve:

\[ \begin{align*}
\text{min} & \quad \lambda \\
\text{s.t} & \quad \lambda x_{31} \geq \zeta_1 x_{11} + \zeta_2 x_{21} + \zeta_3 x_{31} \\
& \quad \lambda x_{32} \geq \zeta_1 x_{12} + \zeta_2 x_{22} + \zeta_3 x_{32} \\
& \quad \lambda x_{33} \geq \zeta_1 x_{13} + \zeta_2 x_{23} + \zeta_3 x_{33} \\
& \quad \lambda x_{34} \geq \zeta_1 x_{14} + \zeta_2 x_{24} + \zeta_3 x_{34} \\
& \quad u_{31} \leq \zeta_1 u_{11} + \zeta_2 u_{21} + \zeta_3 u_{31} \\
& \quad u_{32} \leq \zeta_1 u_{12} + \zeta_2 u_{22} + \zeta_3 u_{32} \\
& \quad \zeta_1 + \zeta_2 + \zeta_3 = 1 \\
& \quad \zeta_1 \geq 0 \quad \zeta_2 \geq 0 \quad \zeta_3 \geq 0
\end{align*} \]

A Detailed Example V

- Or, in matrix form:

\[ \begin{align*}
\text{min} & \quad \lambda \\
\text{s.t} & \quad \lambda \begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \\ x_{34} \end{pmatrix} \geq \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix} \\
& \quad \begin{pmatrix} u_{31} \\ u_{32} \end{pmatrix} \leq \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix} \\
& \quad \zeta_1 + \zeta_2 + \zeta_3 = 1 \\
& \quad \zeta_1 \geq 0 \quad \zeta_2 \geq 0 \quad \zeta_3 \geq 0
\end{align*} \]
That is:

\[
\begin{align*}
\min & \quad \lambda \\
\text{s.t} & \quad \lambda x_3 \geq \zeta' X \\
& \quad u_3 \leq \zeta' U \\
& \quad \zeta' 1 = 1 \\
& \quad \zeta \geq 0
\end{align*}
\]

which we have already seen.

### Measuring Output-Oriented Technical Efficiency II

We can also write this concisely as an optimization problem:

Find the greatest \( \theta \) and the \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_J) \geq 0 \) to

\[
\begin{align*}
\max & \quad \theta \\
\text{s.t} & \quad x_j \geq \zeta' X \\
& \quad \theta u_j \leq \zeta' U \\
& \quad \zeta' 1 = 1 \\
& \quad \zeta \geq 0
\end{align*}
\]

This is another linear program and is also easy to solve.

If we find the solution \( \theta \) to be 1, then system \( j \) is technically efficient in an output-oriented sense.

If the largest \( \theta \) is \( > 1 \) then we have output-oriented technical inefficiency for system \( j \); and the larger the \( \theta \), the greater the degree of inefficiency.

### The Farrell Measures

- The two measures we have discussed (minimum \( \lambda \) and maximum \( \theta \)) are often called the Farrell measures of technical efficiency.
- They measure efficiency by attempting to scale down all inputs by the same proportion \( \lambda \) (for an input orientation) or by attempting to scale up all outputs by the same proportion \( \theta \) (for an output orientation).
- But there is a slight difficulty here.
A Difficulty I

- Suppose in a 2-input case we have identified the feasible region for a given output \( u \) as shaded (This is essentially a piecewise-linear \( u \)-isoquant).
- Suppose we ask about the efficiency of system 1 in an input orientation.
- The minimum scale-back factor for both inputs is \( \lambda \).

A Difficulty II

- The problem here is that although we have correctly identified system 1 as technically inefficient, we have not completely identified the extent of the improvement possible.
- As shown, system 1 could still reduce input 2 (\( x_2 \)) beyond the proportional scale-back identified by \( \lambda x \).

The Russell Measures I

- One way to fix this is to allow each input its own scale-back factor \( \lambda_n \).
- Then instead of minimizing \( \lambda \), we minimize the average scale-back factors:
  \[
  \frac{1}{N} \sum \lambda_n
  \]

The Russell Measures II

- Similar considerations apply to an output orientation, where we maximize the average output-specific scale-up factor:
  \[
  \frac{1}{M} \sum \theta_m
  \]
- These are the Russell measures of technical efficiency.
- This leads to the following pair of problems:

  **Input orientation:**
  \[
  \begin{align*}
  \text{min} & \quad \frac{1}{N} \sum \lambda_n \\
  \text{s.t.} & \quad \lambda_n x_{jn} \geq \zeta' X_n \quad \forall n \\
  & \quad u_{jm} \leq \zeta' U_m \quad \forall m \\
  & \quad \zeta_j \geq 0 \quad \forall j \\
  & \quad \zeta' 1 = 1
  \end{align*}
  \]

  **Output orientation:**
  \[
  \begin{align*}
  \text{max} & \quad \frac{1}{M} \sum \theta_m \\
  \text{s.t.} & \quad x_{jn} \geq \zeta' X_n \quad \forall n \\
  & \quad \theta_m u_{jm} \leq \zeta' U_m \quad \forall m \\
  & \quad \zeta_j \geq 0 \quad \forall j \\
  & \quad \zeta' 1 = 1
  \end{align*}
  \]

  (\( X_n \) is the \( n \)-th column of \( X \); \( U_m \) is the \( m \)-th column of \( U \))
An Application: Bus Transit

- We study systems that provided bus transit only: either conventional motorbus (MB) service, or else demand-responsive (DR) service (or both).
- 217 systems, observed in 1990
- 21 Inputs, including labor of various types: see tables below
- 4 Outputs: Vehicle-miles and trips for DR and MB operations

### Outputs Summary

<table>
<thead>
<tr>
<th>Output</th>
<th>Units</th>
<th>Max</th>
<th>Min</th>
<th>Mean</th>
<th># Nonzero</th>
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<tbody>
<tr>
<td>Vehicle-miles - DR</td>
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<td>102.204</td>
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<td>0.022</td>
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### System Summary

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<th>Public</th>
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<td>DR alone</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>MB alone</td>
<td>111</td>
<td>27</td>
<td>138</td>
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<tr>
<td>MB + DR</td>
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<td>1</td>
<td>79</td>
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<tr>
<td>Total</td>
<td>189</td>
<td>28</td>
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### Inputs Summary I

<table>
<thead>
<tr>
<th>Input</th>
<th>Units</th>
<th>Max</th>
<th>Min</th>
<th>Mean</th>
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<tr>
<td>Average speed - MB</td>
<td>mph</td>
<td>35.580</td>
<td>6.641</td>
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<tr>
<td>Average fleet age</td>
<td>yrs</td>
<td>18.46</td>
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<tr>
<td>MB directional miles</td>
<td>mi</td>
<td>4804</td>
<td>19.6</td>
<td>347.098</td>
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<tr>
<td>Fleet - DR</td>
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<td>10³</td>
<td>.098</td>
<td>.004</td>
<td>79</td>
</tr>
<tr>
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<td>Tires+Materials Cost - MB</td>
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### Inputs Summary II

<table>
<thead>
<tr>
<th>Input</th>
<th>Units</th>
<th>Max</th>
<th>Min</th>
<th>Mean</th>
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</tr>
</thead>
<tbody>
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<td>Labor Hrs - Maint - MB</td>
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<td>Labor Hrs - Admin - DR</td>
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<td>0.936</td>
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<tr>
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<td>0</td>
<td>3.359</td>
<td>35</td>
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</tbody>
</table>

### Solutions

- We measure inefficiency by the Russell measures, allowing for input- and output-specific factors.
- We solve the optimization problems once for each of the 217 systems: this gives us a way to characterize the distribution of inefficiency amongst these operators.

### Results for COTA

Solving the above programs with $j = \text{COTA}$ we find

- COTA is technically efficient in an input-oriented sense
- COTA is technically efficient in an output-oriented sense

### Other Ohio Bus Operators

Other Ohio bus operators are technically efficient in both senses except:

<table>
<thead>
<tr>
<th>System</th>
<th>$\lambda$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canton RTA</td>
<td>0.69</td>
<td>1.33</td>
</tr>
<tr>
<td>Lima-Allen County RTA</td>
<td>0.51</td>
<td>2.39</td>
</tr>
</tbody>
</table>

both of which are quite inefficient.
Allocative Efficiency I

- We have studied input-oriented technical efficiency, the ability of a system to contract inputs without reducing outputs.
- But even if a system is technically efficient, it is still possible that it is utilizing the wrong input mix, relative to the one that allows it to operate at least cost.
- This raises the question of the cost-minimizing input mix, known as allocative efficiency.
- A system is allocatively efficient if it is operating at least cost.

Allocative Efficiency II

- Systems $S_1$ and $S_2$ produce the same output.
- Both face the same input prices (slopes of $c_1$ and $c_2$).
- Both are technically efficient.
- But $S_2$ is allocatively inefficient: by imitating $S_1$ it could save $c_2 - c_1$. 

### Allocative Efficiency III

- We can study allocative efficiency in the same frontier framework as outlined here.
- This requires additional data on input prices.
- The problem is then to minimize input costs, subject to remaining within the feasible production region.
- See references for more on this.

### Software

- The only computational technique needed to solve the efficiency problems discussed here is Linear Programming, which, at least for the types (and dimensions) of problems encountered in practice, is completely straightforward.
- Some statistics packages (e.g., LIMDEP) contain built-in LP modules. There are also dedicated programs (see e.g., the one described in the Coelli *et al.* reference).
- But note that if you want to investigate the distribution of efficiency in an industry, you will need to solve an LP once for each firm/system in your dataset. These LPs are basically the same, differing only in which system is being considered. So it is important to work in an environment that allows you to vary the system under consideration in a flexible way.
- Systems like GAMS are particularly well-suited to this. Another possibility is the commercial MATLAB system, or its free clone, Octave.

### Concluding Remarks

- This has been only an *introduction* to DEA.
- It is possible to formulate alternative models involving different ways of building the frontier, and different assumptions about input/output disposal. See the Färe *et al.* reference for more on this.
- There is also a completely different approach to efficiency, based on statistical estimation of cost and production functions. One advantage of this approach is that it can account for badly measured and/or missing data. A disadvantage is that it is a bit harder to accommodate multiple outputs (though multiple inputs are straightforward). See the Khumbakar-Lovell reference for more on this approach.

### References