Calculating Elasticities

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Suppose we have a demand function

\[ x_1 = x_1^*(p_1, p_2, M, \ldots) \]

where

- \( p_1 \) = own-price of \( x_1 \).
- \( p_2 \) = cross prices (prices of other goods influencing the demand for good \( x_1 \)).
- \( M \) = income.
Suppose that we observe current \((base)\) values for the independent variables:

\[
p_1 = p_1^0 \\
p_2 = p_2^0 \\
M = M^0
\]

And suppose we want to calculate the elasticity of demand with respect to one of the independent variables, call it \(z\). That is, we want:

\[
\eta = \frac{\% \text{ change in quantity demanded}}{\% \text{ change in } z}
\]
There are several ways to calculate elasticities, including:

- The arc elasticity of demand.
- The point elasticity of demand.
- The mid-point-base elasticity of demand.

This note reviews calculation techniques for each of these.
Calculating the Arc Elasticity

Calculating arc elasticities involves the following steps:

1. Plug the base data into the demand function and calculate the quantity demanded $x_1^0$.
2. Vary the quantity of interest $z$ by a small amount $\Delta z$. So we are looking at $z^1 = z^0 + \Delta z$.
3. Plug in the new data into the demand function and calculate the new demand $x_1^1$ in that setting.
4. Calculate the change in demand $\Delta x_1 = x_1^1 - x_1^0$.
5. The arc elasticity of demand is given by:

\[ \eta = \frac{\Delta x_1}{\Delta z} \frac{z^0}{x_1^0} \]
Examples – Setting

Suppose the demand for a good $x_1$ is

$$x_1 = 12 - 0.5p_1 + 0.9p_2 + 0.0001M$$

where:

- $p_1$ is the own-price of $x_1$ ($\$$)
- $p_2$ is the price of some other good (a cross-price) ($\$$)
- $M$ is income ($\$$)

And suppose we have, for our base data

- $p_1^0 = 4$
- $p_2^0 = 8$
- $M^0 = 50,000$
Own-Price Arc-Elasticity of Demand : Example 1

In this case the quantity of interest is $p_1$: we are inquiring about the responsiveness of demand to changes in the own-price. (In other words, in terms of the general notation, $z \equiv p_1$).

1. Plug in the base data and calculate the demand:

$$x_1^0 = 12 - (0.5 \times 4) + (0.9 \times 8) + (0.0001 \times 50000)$$

$$= 12 - 2 + 7.2 + 5$$

$$= 22.2$$

2. Vary $p_1$ by a small amount, say $\Delta p_1 = +0.01$. So we are now looking at the situation where the own-price is $p_1^1 = p_1^0 + \Delta p_1 = 4.01$. 

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1. Calculate the new quantity demanded:

\[ x_1^1 = 12 - (0.5 \times 4.01) + (0.9 \times 8) + (0.0001 \times 50000) \]
\[ = 22.195 \]

2. Calculate \( \Delta x_1 = x_1^1 - x_1^0 = 22.195 - 22.2 = -0.005 \).

3. Elasticity:

\[ \eta = \frac{-0.005}{0.01} \frac{4}{22.2} = 0.09009 \]
Question 1: What About the Order?

- When we compute $\Delta x_1$, why do we take it to be $\Delta x_1 = x^1 - x^0$ (and not the other way round)?
- Answer: we need to compute $\Delta x_1$ and $\Delta p_1$ in the same order from the data.
- We have:
  - Original data: quantity = $x_1^0$, price = $p_1^0$
  - New data: quantity = $x_1^1$, price = $p_1^0 + \Delta p_1$

  - In our calculation, we subtracted the new data from the old: this gave $(p_1^0 + \Delta p_1) - p_1^0 = \Delta p_1$ for the change in price and $x^1 - x^0 = \Delta x_1$ for the change in quantity.

  - If you wanted to do it the other way round, you could take $\Delta x_1 = x^0 - x^1$ but then you would have to take $\Delta p_1 = p_1^0 - (p_1^0 + \Delta p_1) = -\Delta p_1$: note the minus sign.
Cross-Price Arc-Elasticity of Demand: Example

In this case the quantity of interest is \( p_2 \). (So \( z \equiv p_2 \)). We proceed as before:

1. Plug in the base data. We’ve already done this, and found that \( x_1^0 = 22.2 \).

2. Vary \( p_2 \) by a small amount, say \( \Delta p_2 = -0.02 \). So we will be working with a cross price of \( p_2^1 = p_2^0 + \Delta p_2 = 7.98 \).

3. Plug in, to find the new quantity demanded:

\[
\begin{align*}
    x_1^1 &= 12 - (0.5 \times 4) + (0.9 \times 7.98) + (0.0001 \times 50000) \\
          &= 22.182
\end{align*}
\]

4. Compute \( \Delta x_1 = x_1^1 - x_1^0 = 22.182 - 22.2 = -0.018 \).

5. Elasticity:

\[
\eta = \frac{-0.018}{-0.02} \frac{8}{22.2} = 0.32432
\]
Income Arc-Elasticity of Demand: Example

In this case the quantity of interest is $M$ : we ask how responsive the demand for $x_1$ is to changes in income. (So $z \equiv M$).

1. Plug in the base data. We’ve already found that $x_1^0 = 22.2$.
2. Vary $M$ by a small amount, say $+3.00$. So we will be working with $M^1 = M^0 + \Delta M = 50003$.
3. Plug in, to find the new quantity demanded:

$$x_1^1 = 12 - (0.5 \times 4) + (0.9 \times 8) + (0.0001 \times 50003)$$
$$= 22.2003$$

4. Compute $\Delta x = x_1^1 - x_1^0 = 22.2003 - 22.2 = 0.0003$

Compute the elasticity:

$$\eta = \frac{0.0003 \times 50000}{3.00 \times 22.2} = 0.2252252$$
Question 2: What About the Small Change?

- In the examples, could we have chosen another “small change”?
  - Yes, as long as it’s small relative to the base situation, any small change will do.

- But won’t that give us a different answer?
  - Yes it may (if demand is non-linear), but not by much (as long as the changes are really small).

- Aren’t two answers a problem?
  - Not really: what’s happening is that we are using $\Delta x / \Delta z$ (where $z$ is the quantity of interest) to approximate the slope of the demand function at a point. We can live with small differences (approximation errors). If you’re really concerned, consider computing a point elasticity instead, using calculus.
A Different Small Change

Let’s re-calculate the income elasticity of demand with a different small change.

1. We already know that demand at the base point is \( x_1^0 = 22.2 \).
2. Let’s vary income by -1.00, so income = 49,999.
3. Plug in:

\[
x_1^1 = 12 - (0.5 \times 4) + (0.9 \times 8) + (0.0001 \times 49999) \\
= 22.1999
\]

4. Compute \( \Delta x_1 = 22.1999 - 22.2 = -0.0001 \)
5. Elasticity:

\[
\eta = \frac{-0.0001 \times 50000}{-1.00 \times 22.2} \\
= 0.2252252
\]

Note that since demand is linear we get the same answer as before.
Mid-Point Base

- In their book, Call and Holahan recommend using the “mid-point base” instead of the base point as we have done.
- The mid-point base calculation computes the elasticity as:

\[ \eta = \frac{\Delta x}{\Delta z} \frac{(z^0 + z^1)/2}{(x^0 + x^1)/2} \]

- One reason favoring the mid-point base is that it is symmetric: see next examples. This is not so with the arc-elasticity.
- You need to remember that all these “finite change” methods are just approximations to the point elasticity, which involves calculus.
We revisit the own-price elasticity of demand calculation, this time using a mid-point base computation.

1. At the base setting we have $x_1^0 = 22.2$.
2. Vary $p_1$ by $+0.01$.
3. Plug in. We’ve already seen that $x_1^1 = 22.195$.
4. $\Delta x = 22.195 - 22.2 = -0.005$.
5. Mid-point base own-price elasticity:

$$
\eta = \frac{-0.005 \cdot (4.00 + 4.01)/2}{0.01 \cdot (22.2 + 22.195)/2}
= \frac{-0.005 \cdot 4.005}{0.01 \cdot 22.1975}
= -0.09021286
$$
Mid-Point Base: The Other Way Round

We now take the original setting to be $p_1^0 = 4$, $p_2^0 = 8.01$, $M^0 = 50000$. (In other words, our new starting point is the previous final point).

1. At (this) original setting we have already seen that $x_1^0 = 22.195$
2. Vary $p_1$ by $-0.01$, so we will be working with $p_1 = 4$
3. With the new $p_1$ we have already calculated that $x_1^1 = 22.2$
4. Compute $\Delta x = x_1^1 - x_1^0 = 22.2 - 22.195 = 0.005$
5. Mid-point base own-price elasticity:

$$\eta = \frac{0.005 \ (4.00 + 4.01)/2}{-0.01 \ (22.2 + 22.195)/2}$$
$$= -0.09021286$$

as before. This illustrates that the mid-point base computation is symmetric.
Own-Price Point-Elasticity

The point elasticity of demand is defined as:

\[ \eta = \frac{\partial x^*}{\partial z} \frac{z^0}{x^0} \]

where \( z \) is the quantity of interest, and the first term is the (partial) derivative of demand with respect to this quantity.

If you remember your calculus, this may be easier to calculate:

1. Plug in the data and find \( x^0 \), the quantity demand at the original point.
2. Compute the derivative with respect to the quantity of interest.
3. Compute the elasticity using the formula.
Own-Price Point-Elasticity: Example

We compute the own-price point elasticity of demand, recalling that:

\[ x_1 = 12 - 0.5p_1 + 0.9p_2 + 0.0001M \]

1. At the base point, \( p_1^0 = 4, \ p_2^0 = 8, \ M^0 = 50,000 \) we have already found that \( x_1^0 = 22.2 \).
2. The derivative of the demand function with respect to \( p_1 \) is \(-0.5\).
3. The point elasticity is:

\[ \eta = -0.5 \frac{4}{22.2} \]

\[ = -0.09009 \]

Note that because demand is linear, this agrees with the arc-elasticity (but not with the mid-point base case, though they’re not far apart).
The income *point* elasticity of demand:

1. At the base point $x_1^0 = 22.2$.
2. The derivative of the demand function with respect to income is 0.0001.
3. The point elasticity is:

$$ \eta = 0.0001 \frac{50000}{22.2} $$

$$ = 0.225\ 225\ 2 $$
What Should I Do?

All these ways of computing elasticity — which should I use (in an exam, for example)?

- The quick answer is that you can use any of them.
- If you’re confident of your calculus, then it’s probably best to compute the point elasticity: that’s the one you’ll usually find in the literature.
- Otherwise, it’s your choice. Many people find the arc elasticity slightly less work than the mid-point base calculation.
- But remember, all the finite change methods are just approximations to the calculus-based point elasticity.